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Tabulation of the Function $\psi(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$

Introduction. The function $\phi(x)$ defined by

$$\phi(x) = \int_1^x \frac{dt}{t} \log|1 + t| \tag{1}$$

occurs quite frequently in problems in Quantum Electrodynamics, as for example problems on vacuum polarization [1], scattering of light by light [2], and similar fourth order processes [3].

For real values of x, the function $\phi(x)$ has been tabulated by Mitchell [4]. His tabulated function is $\phi(0) - \phi(x)$, rather than our $\phi(x)$ defined by (1). As in most literature he defines it as minus the integral from 0 to x. We shall however stick to the definition (1), which has already appeared in some of the literature [1].

The integrals appearing in the above mentioned problems cannot always be expressed in terms of $\phi(x)$ with the argument x real. In many cases there appear $\phi(x)$ with a complex x of absolute value 1. This happens when the integrals in question are expressed as combinations of $\phi\left(\frac{1-\delta}{1+\delta}\right)$ and $\phi\left(-\frac{1-\delta}{1+\delta}\right)$ where δ^2 is to take all values from $-\infty$ to $+\infty$. When δ^2 is negative, the complex argument $1 - \delta/1 + \delta$ will have absolute value 1, and the problem reduces to that of finding $\phi(e^{i\theta})$. On use of the expansion

$$\phi(x) = \sum_1^{\infty} \frac{(-)^{n+1}x^n}{n^2} - \frac{\pi^2}{12} \tag{2}$$

the latter function can be expressed as

$$\phi(e^{i\theta}) = -\theta^2/4 + i\psi(\theta) \tag{3}$$

where

$$\psi(\theta) = \sum_1^{\infty} \frac{\sin n\theta}{n^2}. \tag{4}$$

It was however shown by Kummer [5] that for complex x the imaginary part of $\phi(x)$ can be expressed in terms of $\psi(\theta)$ defined by equation (4).

The function $\psi(\theta)$, the Spence function, was tabulated by Clausen [6] for $\theta = 1, 2, \cdots, 180°$. We find that these given intervals are not enough for easy computation of the function for any argument.

A tabulation of the function $\psi(\theta)$ for enough intervals will be useful for physicists to carry out explicit numerical evaluations of their calculations for comparison with recent accurate experiment, which is necessary for the development of the theory. Therefore we found it worth while tabulating the function, given by Clausen for 180 intervals, for more intervals in such a way that the function can be easily computed for any value of the argument θ.

Some properties of the function $\psi(\theta)$. The series $\sum_1^\infty \frac{\sin n\theta}{n^2}$ is slowly convergent and the function is best computed by a numerical evaluation of the integral

$$\psi(\theta) = -\int_0^\theta \log\left(2\sin\frac{t}{2}\right)dt. \tag{5}$$

The function is zero at $\theta = 0$, $\theta = \pi$ and has a maximum between these two limits at $\theta = \pi/3$. For $\theta < \pi/2$ the function can be expressed by the expansion

$$\psi(\theta) = -\theta\log|\theta| + \theta + \sum_1^\infty \frac{(-)^{n-1}}{(2n)!}B_{2n}\frac{\theta^{2n+1}}{2n(2n+1)}. \tag{6}$$

For $\pi/2 < \theta < \pi$ the function can be expanded by use of the recurrence relation

$$\psi(\pi - \theta) = \psi(\theta) - \tfrac{1}{2}\psi(2\theta) \tag{7}$$

and is given by

$$\psi(\pi - \theta) = \theta\log 2 - \sum_1^\infty \frac{(-)^{n-1}}{(2n)!}B_{2n}(2^{2n} - 1)\frac{\theta^{2n+1}}{2n(2n+1)} \tag{8}$$

which shows that the function is linear in θ for θ near π.

Method of calculation. We start by evaluating the function from $\theta = \pi$ to $\theta = \pi/2$. The tabulated function $\log_{10}(\sin\frac{1}{2}t)$ to 7 figures is used to obtain the value of the integrand in (5), at 540 intervals: $\theta_0, \theta_1, \cdots$. We have then used the known expression

$$\psi(\theta_n) = w[\tfrac{1}{2}f(\theta_0) + f(\theta_1) + \cdots + f(\theta_{n-1}) + \tfrac{1}{2}f(\theta_n)] - \frac{w^2}{12}[f'(\theta_n) - f'(\theta_0)] \tag{9}$$

to compute $\psi(\theta_n)$. Here $f(\theta_i)$ is the integrand and w is the interval $\pi/1080$. The remainder containing w^4 and higher powers was found not to affect the result expressed by 7 figures.

The last tabulated value at $\theta = \pi/2$ was found to agree with the known Catalan's number, $\sum_1^\infty \frac{(-)^n}{(2n+1)^2}$, which was a good check on this first part of the work.

The value of $\psi(\theta)$ from $\theta = \pi/2$ to $\theta = 0$ will then be computed from the already calculated values on use of the relation $\psi(\theta) = \psi(\pi - \theta) + \frac{1}{2}\psi(2\theta)$. On

reaching $\theta = \pi/4$ we had to start using newly calculated values of $\psi(\theta)$ to complete the table until $\theta = 0$. For example the value at $\theta = 44°$ is $\psi(136°) + \frac{1}{2}\psi(88°)$. The value at $\theta = 136°$ is contained in the first part of the work while the value at $\theta = 88°$ must be obtained from $\psi(92°) + \frac{1}{2}\psi(176°)$, which is contained in the beginning of the second part of the work.

The function was found to have the required maximum at $\theta = \pi/3$ and was further checked by directly evaluating it, using the expansion (6) for several values of θ.

The whole work has been carried out keeping the number of decimals 7, but it was found safer to tabulate it finally to 6 places of decimals which is enough for most purposes.

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x	$\psi(x)$	δ
0° 00′	0.000000	
10	0.019897	19897
20	0.035761	15864
30	0.050103	14342
40	0.063457	13354
50	0.076075	12618
1° 00	0.088108	12033
10	0.099654	11546
20	0.110783	11129
30	0.121547	10764
40	0.131988	10441
50	0.142137	10149
2° 00	0.152022	9885
10	0.161663	9641
20	0.171081	9418
30	0.180291	9210
40	0.189307	9016
50	0.198141	8834
3° 00	0.206803	8662
10	0.215304	8501
20	0.223652	8348
30	0.231855	8203
40	0.239919	8064
50	0.247851	7932
4° 00	0.255656	7805
10	0.263340	7684
20	0.270908	7568
30	0.278364	7456
40	0.285712	7348
50	0.292956	7244
5° 00	0.300100	7144
10	0.307147	7047
20	0.314101	6954
30	0.320963	6862
40	0.327738	6775
50	0.334427	6689
6° 00	0.341032	6605
10	0.347558	6526
20	0.354004	6446
30	0.360374	6370
40	0.366670	6296
50	0.372893	6223
7° 00	0.379045	6152
10	0.385128	6083
20	0.391143	6015
30	0.397092	5949
40	0.402977	5885
50	0.408799	5822
8° 00	0.414558	5759
10	0.420258	5700
20	0.425898	5640
30	0.431480	5582
40	0.437005	5525
50	0.442474	5469
9° 00	0.447888	5414
10	0.453249	5361
20	0.458557	5308
30	0.463813	5256
40	0.469018	5205
50	0.474173	5155
10° 00	0.479279	5106

x	$\psi(x)$	δ
10° 00′	0.479279	
10	0.484336	5057
20	0.489346	5010
30	0.494309	4963
40	0.499227	4918
50	0.504098	4871
11° 00	0.508925	4827
10	0.513709	4784
20	0.518449	4740
30	0.523146	4697
40	0.527801	4655
50	0.532415	4614
12° 00	0.536988	4573
10	0.541521	4533
20	0.546014	4493
30	0.550468	4454
40	0.554883	4415
50	0.559261	4378
13° 00	0.563600	4339
10	0.567903	4303
20	0.572168	4265
30	0.576398	4230
40	0.580592	4194
50	0.584750	4158
14° 00	0.588874	4124
10	0.592963	4089
20	0.597018	4055
30	0.601040	4022
40	0.605028	3988
50	0.608983	3955
15° 00	0.612906	3923
10	0.616797	3891
20	0.620656	3859
30	0.624483	3827
40	0.628280	3797
50	0.632045	3765
16° 00	0.635781	3736
10	0.639486	3705
20	0.643161	3675
30	0.646807	3646
40	0.650424	3617
50	0.654012	3588
17° 00	0.657571	3559
10	0.661102	3531
20	0.664605	3503
30	0.668080	3475
40	0.671527	3447
50	0.674948	3421
18° 00	0.678341	3393
10	0.681708	3367
20	0.685048	3340
30	0.688362	3314
40	0.691650	3288
50	0.694912	3262
19° 00	0.698149	3237
10	0.701361	3212
20	0.704547	3186
30	0.707709	3162
40	0.710846	3137
50	0.713958	3112
20° 00	0.717047	3089

x	$\psi(x)$	δ
20° 00′	0.717047	
10	0.720111	3064
20	0.723152	3041
30	0.726169	3017
40	0.729163	2994
50	0.732133	2970
21° 00	0.735080	2947
10	0.738005	2925
20	0.740907	2902
30	0.743786	2879
40	0.746644	2858
50	0.749479	2835
22° 00	0.752292	2813
10	0.755083	2791
20	0.757853	2770
30	0.760601	2748
40	0.763328	2727
50	0.766034	2706
23° 00	0.768719	2685
10	0.771384	2665
20	0.774027	2643
30	0.776650	2623
40	0.779253	2603
50	0.781835	2582
24° 00	0.784398	2563
10	0.786941	2543
20	0.789463	2522
30	0.791967	2504
40	0.794450	2483
50	0.796915	2465
25° 00	0.799360	2445
10	0.801786	2426
20	0.804193	2407
30	0.806581	2388
40	0.808951	2370
50	0.811302	2351
26° 00	0.813635	2333
10	0.815949	2314
20	0.818245	2296
30	0.820523	2278
40	0.822783	2260
50	0.825025	2242
27° 00	0.827249	2224
10	0.829456	2207
20	0.831646	2190
30	0.833817	2171
40	0.835972	2155
50	0.838109	2137
28° 00	0.840230	2121
10	0.842333	2103
20	0.844420	2087
30	0.846489	2069
40	0.848542	2053
50	0.850579	2037
29° 00	0.852599	2020
10	0.854603	2004
20	0.856590	1987
30	0.858561	1971
40	0.860517	1956
50	0.862456	1939
30° 00	0.864379	1923

x	$\psi(x)$	δ
30° 00′	0.864379	
		1908
10	0.866287	
		1892
20	0.868179	
		1876
30	0.870055	
		1861
40	0.871916	
		1845
50	0.873761	
		1830
31° 00	0.875591	
		1814
10	0.877405	
		1800
20	0.879205	
		1784
30	0.880989	
		1770
40	0.882759	
		1755
50	0.884514	
		1739
32° 00	0.886253	
		1725
10	0.887978	
		1710
20	0.889688	
		1696
30	0.891384	
		1681
40	0.893065	
		1667
50	0.894732	
		1652
33° 00	0.896384	
		1638
10	0.898022	
		1624
20	0.899646	
		1610
30	0.901256	
		1596
40	0.902852	
		1581
50	0.904433	
		1568
34° 00	0.906001	
		1554
10	0.907555	
		1540
20	0.909095	
		1527
30	0.910622	
		1513
40	0.912135	
		1499
50	0.913634	
		1486
35° 00	0.915120	
		1472
10	0.916592	
		1459
20	0.918051	
		1446
30	0.919497	
		1432
40	0.920929	
		1420
50	0.922349	
		1406
36° 00	0.923755	
		1393
10	0.925148	
		1381
20	0.926529	
		1367
30	0.927896	
		1355
40	0.929251	
		1342
50	0.930593	
		1329
37° 00	0.931922	
		1316
10	0.933238	
		1304
20	0.934542	
		1292
30	0.935834	
		1278
40	0.937112	
		1267
50	0.938379	
		1254
38° 00	0.939633	
		1242
10	0.940875	
		1230
20	0.942105	
		1217
30	0.943322	
		1206
40	0.944528	
		1193
50	0.945721	
		1181
39° 00	0.946902	
		1170
10	0.948072	
		1157
20	0.949229	
		1146
30	0.950375	
		1133
40	0.951508	
		1123
50	0.952631	
		1110
40° 00	0.953741	

x	$\psi(x)$	δ
40° 00′	0.953741	
		1099
10	0.954840	
		1087
20	0.955927	
		1076
30	0.957003	
		1064
40	0.958067	
		1053
50	0.959120	
		1041
41° 00	0.960161	
		1031
10	0.961192	
		1018
20	0.962210	
		1008
30	0.963218	
		997
40	0.964215	
		985
50	0.965200	
		974
42° 00	0.966174	
		964
10	0.967138	
		952
20	0.968090	
		941
30	0.969031	
		931
40	0.969962	
		920
50	0.970882	
		908
43° 00	0.971790	
		899
10	0.972689	
		887
20	0.973576	
		877
30	0.974453	
		866
40	0.975319	
		856
50	0.976175	
		845
44° 00	0.977020	
		834
10	0.977854	
		825
20	0.978679	
		814
30	0.979493	
		803
40	0.980296	
		793
50	0.981089	
		783
45° 00	0.981872	
		773
10	0.982645	
		762
20	0.983407	
		753
30	0.984160	
		742
40	0.984902	
		733
50	0.985635	
		722
46° 00	0.986357	
		712
10	0.987069	
		703
20	0.987772	
		692
30	0.988464	
		683
40	0.989147	
		673
50	0.989820	
		663
47° 00	0.990483	
		653
10	0.991136	
		644
20	0.991780	
		634
30	0.992414	
		624
40	0.993038	
		615
50	0.993653	
		605
48° 00	0.994258	
		596
10	0.994854	
		586
20	0.995440	
		577
30	0.996017	
		567
40	0.996584	
		559
50	0.997143	
		548
49° 00	0.997691	
		540
10	0.998231	
		530
20	0.998761	
		521
30	0.999282	
		512
40	0.999794	
		503
50	1.000297	
		494
50° 00	1.000791	

x	$\psi(x)$	δ
50° 00′	1.000791	
		484
10	1.001275	
		476
20	1.001751	
		466
30	1.002217	
		458
40	1.002675	
		448
50	1.003123	
		440
51° 00	1.003563	
		431
10	1.003994	
		422
20	1.004416	
		413
30	1.004829	
		405
40	1.005234	
		395
50	1.005629	
		387
52° 00	1.006016	
		379
10	1.006395	
		369
20	1.006764	
		361
30	1.007125	
		353
40	1.007478	
		344
50	1.007822	
		335
53° 00	1.008157	
		327
10	1.008484	
		318
20	1.008802	
		311
30	1.009113	
		301
40	1.009414	
		294
50	1.009708	
		284
54° 00	1.009992	
		277
10	1.010269	
		268
20	1.010537	
		261
30	1.010798	
		252
40	1.011050	
		243
50	1.011293	
		236
55° 00	1.011529	
		227
10	1.011756	
		220
20	1.011976	
		211
30	1.012187	
		203
40	1.012390	
		195
50	1.012585	
		188
56° 00	1.012773	
		179
10	1.012952	
		171
20	1.013123	
		164
30	1.013287	
		155
40	1.013442	
		148
50	1.013590	
		140
57° 00	1.013730	
		132
10	1.013862	
		125
20	1.013987	
		116
30	1.014103	
		109
40	1.014212	
		101
50	1.014313	
		94
58° 00	1.014407	
		86
10	1.014493	
		78
20	1.014571	
		71
30	1.014642	
		63
40	1.014705	
		56
50	1.014761	
		48
59° 00	1.014809	
		40
10	1.014849	
		34
20	1.014883	
		26
30	1.014909	
		18
40	1.014927	
		11
50	1.014938	
		4
60° 00	1.014942	

x	$\psi(x)$	$-\delta$
60° 00′	1.014942	
10	1.014938	4
20	1.014927	11
30	1.014909	18
40	1.014883	26
50	1.014851	32
61° 00	1.014811	40
10	1.014763	48
20	1.014709	54
30	1.014648	61
40	1.014579	69
50	1.014504	75
62° 00	1.014421	83
10	1.014331	90
20	1.014235	96
30	1.014131	104
40	1.014020	111
50	1.013902	118
63° 00	1.013778	124
10	1.013646	132
20	1.013508	138
30	1.013363	145
40	1.013211	152
50	1.013052	159
64° 00	1.012886	166
10	1.012714	172
20	1.012535	179
30	1.012349	186
40	1.012156	193
50	1.011957	199
65° 00	1.011751	206
10	1.011538	213
20	1.011319	219
30	1.011093	226
40	1.010861	232
50	1.010622	239
66° 00	1.010376	246
10	1.010124	252
20	1.009866	258
30	1.009601	265
40	1.009329	272
50	1.009052	277
67° 00	1.008767	285
10	1.008477	290
20	1.008180	297
30	1.007876	304
40	1.007566	310
50	1.007250	316
68° 00	1.006928	322
10	1.006599	329
20	1.006265	334
30	1.005924	341
40	1.005576	348
50	1.005223	353
69° 00	1.004863	360
10	1.004497	366
20	1.004125	372
30	1.003747	378
40	1.003363	384
50	1.002973	390
70° 00	1.002576	397

x	$\psi(x)$	$-\delta$
70° 00′	1.002576	
10	1.002174	402
20	1.001766	408
30	1.001351	415
40	1.000931	420
50	1.000505	426
71° 00	1.000072	433
10	0.999634	438
20	0.999190	444
30	0.998740	450
40	0.998284	456
50	0.997822	462
72° 00	0.997355	467
10	0.996881	474
20	0.996402	479
30	0.995917	485
40	0.995426	491
50	0.994930	496
73° 00	0.994427	503
10	0.993919	508
20	0.993406	513
30	0.992886	520
40	0.992361	525
50	0.991831	530
74° 00	0.991294	537
10	0.990752	542
20	0.990205	547
30	0.989652	553
40	0.989093	559
50	0.988529	564
75° 00	0.987959	570
10	0.987384	575
20	0.986803	581
30	0.986217	586
40	0.985625	592
50	0.985028	597
76° 00	0.984425	603
10	0.983817	608
20	0.983204	613
30	0.982585	619
40	0.981961	624
50	0.981331	630
77° 00	0.980696	635
10	0.980056	640
20	0.979411	645
30	0.978760	651
40	0.978104	656
50	0.977443	661
78° 00	0.976776	667
10	0.976104	672
20	0.975427	677
30	0.974745	682
40	0.974058	687
50	0.973366	692
79° 00	0.972668	698
10	0.971965	703
20	0.971257	708
30	0.970544	713
40	0.969826	718
50	0.969103	723
80° 00	0.968375	728

x	$\psi(x)$	$-\delta$
80° 00′	0.968375	
10	0.967642	733
20	0.966903	739
30	0.966160	743
40	0.965412	748
50	0.964658	754
81° 00	0.963900	758
10	0.963137	763
20	0.962369	768
30	0.961596	773
40	0.960818	778
50	0.960035	783
82° 00	0.959247	788
10	0.958455	792
20	0.957657	798
30	0.956855	802
40	0.956048	807
50	0.955236	812
83° 00	0.954419	817
10	0.953598	821
20	0.952771	827
30	0.951940	831
40	0.951105	835
50	0.950264	841
84° 00	0.949419	845
10	0.948569	850
20	0.947714	855
30	0.946855	859
40	0.945991	864
50	0.945123	868
85° 00	0.944249	874
10	0.943372	877
20	0.942489	883
30	0.941602	887
40	0.940711	891
50	0.939814	897
86° 00	0.938914	900
10	0.938008	906
20	0.937099	909
30	0.936184	915
40	0.935266	918
50	0.934342	924
87° 00	0.933415	927
10	0.932482	933
20	0.931546	936
30	0.930605	941
40	0.929659	946
50	0.928709	950
88° 00	0.927755	954
10	0.926796	959
20	0.925833	963
30	0.924866	967
40	0.923894	972
50	0.922918	976
89° 00	0.921938	980
10	0.920953	985
20	0.919964	989
30	0.918971	993
40	0.917973	998
50	0.916972	1001
90° 00	0.915966	1006

x	$\psi(x)$	$-\delta$
90° 00′	0.915966	
10	0.914955	1011
20	0.913941	1014
30	0.912922	1019
40	0.911899	1023
50	0.910872	1027
91° 00	0.909841	1031
10	0.908806	1035
20	0.907766	1040
30	0.906722	1044
40	0.905675	1047
50	0.904623	1052
92° 00	0.903567	1056
10	0.902507	1060
20	0.901443	1064
30	0.900374	1069
40	0.899302	1072
50	0.898226	1076
93° 00	0.897145	1081
10	0.896061	1084
20	0.894973	1088
30	0.893880	1093
40	0.892784	1096
50	0.891684	1100
94° 00	0.890580	1104
10	0.889471	1109
20	0.888359	1112
30	0.887243	1116
40	0.886123	1120
50	0.884999	1124
95° 00	0.883872	1127
10	0.882740	1132
20	0.881605	1135
30	0.880465	1140
40	0.879322	1143
50	0.878175	1147
96° 00	0.877024	1151
10	0.875870	1154
20	0.874711	1159
30	0.873549	1162
40	0.872383	1166
50	0.871213	1170
97° 00	0.870040	1173
10	0.868862	1178
20	0.867681	1181
30	0.866497	1184
40	0.865308	1189
50	0.864116	1192
98° 00	0.862920	1196
10	0.861721	1199
20	0.860518	1203
30	0.859311	1207
40	0.858100	1211
50	0.856886	1214
99° 00	0.855668	1218
10	0.854447	1221
20	0.853222	1225
30	0.851994	1228
40	0.850762	1232
50	0.849526	1236
100° 00	0.848287	1239

x	$\psi(x)$	$-\delta$
100° 00′	0.848287	
10	0.847044	1243
20	0.845797	1247
30	0.844548	1249
40	0.843294	1254
50	0.842037	1257
101° 00	0.840777	1260
10	0.839513	1264
20	0.838246	1267
30	0.836975	1271
40	0.835701	1274
50	0.834423	1278
102° 00	0.833142	1281
10	0.831857	1285
20	0.830569	1288
30	0.829278	1291
40	0.827983	1295
50	0.826685	1298
103° 00	0.825383	1302
10	0.824078	1305
20	0.822770	1308
30	0.821458	1312
40	0.820143	1315
50	0.818825	1318
104° 00	0.817503	1322
10	0.816179	1324
20	0.814850	1329
30	0.813519	1331
40	0.812184	1335
50	0.810846	1338
105° 00	0.809505	1341
10	0.808160	1345
20	0.806813	1347
30	0.805462	1351
40	0.804107	1355
50	0.802750	1357
106° 00	0.801389	1361
10	0.800025	1364
20	0.798658	1367
30	0.797288	1370
40	0.795915	1373
50	0.794538	1377
107° 00	0.793159	1379
10	0.791776	1383
20	0.790390	1386
30	0.789001	1389
40	0.787609	1392
50	0.786214	1395
108° 00	0.784816	1398
10	0.783414	1402
20	0.782010	1404
30	0.780603	1407
40	0.779192	1411
50	0.777779	1413
109° 00	0.776362	1417
10	0.774943	1419
20	0.773520	1423
30	0.772094	1426
40	0.770666	1428
50	0.769234	1432
110° 00	0.767800	1434

x	$\psi(x)$	$-\delta$
110° 00′	0.767800	
10	0.766362	1438
20	0.764922	1440
30	0.763478	1444
40	0.762032	1446
50	0.760583	1449
111° 00	0.759131	1452
10	0.757676	1455
20	0.756218	1458
30	0.754757	1461
40	0.753293	1464
50	0.751826	1467
112° 00	0.750357	1469
10	0.748885	1472
20	0.747410	1475
30	0.745932	1478
40	0.744451	1481
50	0.742967	1484
113° 00	0.741481	1486
10	0.739991	1490
20	0.738499	1492
30	0.737004	1495
40	0.735507	1497
50	0.734006	1501
114° 00	0.732503	1503
10	0.730997	1506
20	0.729489	1508
30	0.727978	1511
40	0.726463	1515
50	0.724947	1516
115° 00	0.723427	1520
10	0.721905	1522
20	0.720380	1525
30	0.718853	1527
40	0.717322	1531
50	0.715790	1532
116° 00	0.714254	1536
10	0.712716	1538
20	0.711175	1541
30	0.709632	1543
40	0.708086	1546
50	0.706537	1549
117° 00	0.704986	1551
10	0.703432	1554
20	0.701875	1557
30	0.700316	1559
40	0.698755	1561
50	0.697191	1564
118° 00	0.695624	1567
10	0.694055	1569
20	0.692483	1572
30	0.690909	1574
40	0.689332	1577
50	0.687753	1579
119° 00	0.686171	1582
10	0.684586	1585
20	0.683000	1586
30	0.681410	1590
40	0.679819	1591
50	0.678224	1595
120° 00	0.676628	1596

x	$\psi(x)$	$-\delta$
120° 00′	0.676628	
10	0.675029	1599
20	0.673427	1602
30	0.671823	1604
40	0.670217	1606
50	0.668608	1609
121° 00	0.666997	1611
10	0.665383	1614
20	0.663767	1616
30	0.662149	1618
40	0.660528	1621
50	0.658905	1623
122° 00	0.657280	1625
10	0.655652	1628
20	0.654022	1630
30	0.652390	1632
40	0.650755	1635
50	0.649118	1637
123° 00	0.647478	1640
10	0.645837	1641
20	0.644193	1644
30	0.642547	1646
40	0.640898	1649
50	0.639247	1651
124° 00	0.637594	1653
10	0.635939	1655
20	0.634281	1658
30	0.632622	1659
40	0.630960	1662
50	0.629295	1665
125° 00	0.627629	1666
10	0.625960	1669
20	0.624290	1670
30	0.622617	1673
40	0.620941	1676
50	0.619264	1677
126° 00	0.617585	1679
10	0.615903	1682
20	0.614219	1684
30	0.612533	1686
40	0.610845	1688
50	0.609155	1690
127° 00	0.607463	1692
10	0.605768	1695
20	0.604072	1696
30	0.602373	1699
40	0.600672	1701
50	0.598969	1703
128° 00	0.597265	1704
10	0.595558	1707
20	0.593849	1709
30	0.592138	1711
40	0.590424	1714
50	0.588709	1715
129° 00	0.586992	1717
10	0.585273	1719
20	0.583552	1721
30	0.581829	1723
40	0.580104	1725
50	0.578376	1728
130° 00	0.576647	1729

x	$\psi(x)$	$-\delta$
130° 00′	0.576647	
10	0.574916	1731
20	0.573183	1733
30	0.571448	1735
40	0.569711	1737
50	0.567972	1739
131° 00	0.566231	1741
10	0.564489	1742
20	0.562744	1745
30	0.560997	1747
40	0.559249	1748
50	0.557498	1751
132° 00	0.555746	1752
10	0.553992	1754
20	0.552236	1756
30	0.550478	1758
40	0.548718	1760
50	0.546956	1762
133° 00	0.545193	1763
10	0.543427	1766
20	0.541660	1767
30	0.539891	1769
40	0.538120	1771
50	0.536348	1772
134° 00	0.534573	1775
10	0.532797	1776
20	0.531019	1778
30	0.529239	1780
40	0.527458	1781
50	0.525674	1784
135° 00	0.523889	1785
10	0.522102	1787
20	0.520314	1788
30	0.518524	1790
40	0.516731	1793
50	0.514938	1793
136° 00	0.513142	1796
10	0.511345	1797
20	0.509546	1799
30	0.507745	1801
40	0.505943	1802
50	0.504139	1804
137° 00	0.502334	1805
10	0.500526	1808
20	0.498717	1809
30	0.496907	1810
40	0.495094	1813
50	0.493280	1814
138° 00	0.491465	1815
10	0.489648	1817
20	0.487829	1819
30	0.486008	1821
40	0.484186	1822
50	0.482363	1823
139° 00	0.480538	1825
10	0.478711	1827
20	0.476883	1828
30	0.475053	1830
40	0.473221	1832
50	0.471388	1833
140° 00	0.469554	1834

x	$\psi(x)$	$-\delta$
140° 00′	0.469554	
10	0.467717	1837
20	0.465880	1837
30	0.464041	1839
40	0.462200	1841
50	0.460358	1842
141° 00	0.458514	1844
10	0.456669	1845
20	0.454822	1847
30	0.452974	1848
40	0.451124	1850
50	0.449273	1851
142° 00	0.447421	1852
10	0.445567	1854
20	0.443711	1856
30	0.441854	1857
40	0.439996	1858
50	0.438136	1860
143° 00	0.436275	1861
10	0.434412	1863
20	0.432548	1864
30	0.430683	1865
40	0.428816	1867
50	0.426947	1869
144° 00	0.425078	1869
10	0.423207	1871
20	0.421334	1873
30	0.419461	1873
40	0.417586	1875
50	0.415709	1877
145° 00	0.413831	1878
10	0.411952	1879
20	0.410072	1880
30	0.408190	1882
40	0.406307	1883
50	0.404423	1884
146° 00	0.402537	1886
10	0.400650	1887
20	0.398762	1888
30	0.396872	1890
40	0.394982	1890
50	0.393090	1892
147° 00	0.391196	1894
10	0.389302	1894
20	0.387406	1896
30	0.385509	1897
40	0.383610	1899
50	0.381711	1899
148° 00	0.379810	1901
10	0.377908	1902
20	0.376005	1903
30	0.374101	1904
40	0.372195	1906
50	0.370288	1907
149° 00	0.368380	1908
10	0.366471	1909
20	0.364561	1910
30	0.362649	1912
40	0.360737	1912
50	0.358823	1914
150° 00	0.356908	1915

x	$\psi(x)$	$-\delta$
150° 00′	0.356908	1916
10	0.354992	1917
20	0.353075	1918
30	0.351157	1919
40	0.349238	1921
50	0.347317	1921
151° 00	0.345396	1923
10	0.343473	1924
20	0.341549	1925
30	0.339624	1926
40	0.337698	1927
50	0.335771	1928
152° 00	0.333843	1929
10	0.331914	1930
20	0.329984	1931
30	0.328053	1932
40	0.326121	1934
50	0.324187	1934
153° 00	0.322253	1935
10	0.320318	1936
20	0.318382	1938
30	0.316444	1938
40	0.314506	1939
50	0.312567	1941
154° 00	0.310626	1941
10	0.308685	1942
20	0.306743	1943
30	0.304800	1944
40	0.302856	1945
50	0.300911	1946
155° 00	0.298965	1947
10	0.297018	1948
20	0.295070	1949
30	0.293121	1950
40	0.291171	1951
50	0.289220	1951
156° 00	0.287269	1953
10	0.285316	1953
20	0.283363	1954
30	0.281409	1955
40	0.279454	1956
50	0.277498	1957
157° 00	0.275541	1958
10	0.273583	1958
20	0.271625	1960
30	0.269665	1960
40	0.267705	1961
50	0.265744	1962
158° 00	0.263782	1963
10	0.261819	1963
20	0.259856	1965
30	0.257891	1965
40	0.255926	1966
50	0.253960	1967
159° 00	0.251993	1967
10	0.250026	1969
20	0.248057	1969
30	0.246088	1970
40	0.244118	1970
50	0.242148	1972
160° 00	0.240176	

x	$\psi(x)$	$-\delta$
160° 00′	0.240176	1972
10	0.238204	1973
20	0.236231	1973
30	0.234258	1975
40	0.232283	1975
50	0.230308	1975
161° 00	0.228333	1977
10	0.226356	1977
20	0.224379	1978
30	0.222401	1978
40	0.220423	1980
50	0.218443	1980
162° 00	0.216463	1980
10	0.214483	1981
20	0.212502	1982
30	0.210520	1983
40	0.208537	1983
50	0.206554	1984
163° 00	0.204570	1984
10	0.202586	1985
20	0.200601	1986
30	0.198615	1986
40	0.196629	1987
50	0.194642	1988
164° 00	0.192654	1988
10	0.190666	1989
20	0.188677	1989
30	0.186688	1990
40	0.184698	1990
50	0.182708	1991
165° 00	0.180717	1992
10	0.178725	1992
20	0.176733	1993
30	0.174740	1993
40	0.172747	1994
50	0.170753	1994
166° 00	0.168759	1995
10	0.166764	1995
20	0.164769	1996
30	0.162773	1996
40	0.160777	1997
50	0.158780	1997
167° 00	0.156783	1998
10	0.154785	1998
20	0.152787	1999
30	0.150788	1999
40	0.148789	2000
50	0.146789	2000
168° 00	0.144789	2000
10	0.142789	2001
20	0.140788	2002
30	0.138786	2002
40	0.136784	2002
50	0.134782	2002
169° 00	0.132780	2003
10	0.130777	2004
20	0.128773	2004
30	0.126769	2004
40	0.124765	2005
50	0.122760	2005
170° 00	0.120755	

x	$\psi(x)$	$-\delta$
170° 00′	0.120755	2005
10	0.118750	2006
20	0.116744	2006
30	0.114738	2006
40	0.112732	2007
50	0.110725	2007
171° 00	0.108718	2008
10	0.106710	2008
20	0.104702	2008
30	0.102694	2008
40	0.100686	2009
50	0.098677	2009
172° 00	0.096668	2009
10	0.094659	2010
20	0.092649	2010
30	0.090639	2010
40	0.088629	2010
50	0.086619	2011
173° 00	0.084608	2011
10	0.082597	2011
20	0.080586	2012
30	0.078574	2012
40	0.076562	2011
50	0.074551	2013
174° 00	0.072538	2012
10	0.070526	2013
20	0.068513	2013
30	0.066500	2013
40	0.064487	2013
50	0.062474	2013
175° 00	0.060461	2014
10	0.058447	2014
20	0.056433	2014
30	0.054419	2014
40	0.052405	2014
50	0.050391	2014
176° 00	0.048377	2015
10	0.046362	2015
20	0.044347	2015
30	0.042332	2015
40	0.040317	2015
50	0.038302	2015
177° 00	0.036287	2015
10	0.034272	2016
20	0.032256	2015
30	0.030241	2016
40	0.028225	2016
50	0.026209	2015
178° 00	0.024194	2016
10	0.022178	2016
20	0.020162	2016
30	0.018146	2016
40	0.016130	2016
50	0.014114	2017
179° 00	0.012097	2016
10	0.010081	2016
20	0.008065	2016
30	0.006049	2016
40	0.004033	2016
50	0.002016	2016
180° 00	0.000000	

A Modified Chebyshev-Everett Interpolation Formula

1. Introduction. It is a well-known property of the Chebyshev polynomials that, of all polynomials with leading coefficient unity, they possess the smallest absolute upper bound when the argument is allowed to vary between their limits of orthogonality. This property suggests the use of such polynomials as a means of interpolation. As is explained in Kopal's recent book on numerical analysis [1], a power series rearranged as a series of Chebyshev polynomials is reduced to an economic form, insofar as the "maximum" accuracy is attained with a very small number of terms. We use Kopal's notation [1] here.

The purpose of this note is to rearrange the Everett interpolation formula,

$$f(m) = f(0) + m\Delta_1 + E_0^{[1]}\Delta_0^{[1]} + E_1^{[1]}\Delta_1^{[1]} + \cdots + E_0^{(2n)}\Delta_0^{(2n)} + E_1^{(2n)}\Delta_1^{(2n)}$$

as a series of suitable Chebyshev polynomials, thereby reducing considerably the number of terms depending on m, required to give a prescribed accuracy, while retaining the use of even order differences only. Following Miller [2] the polynomials $C_{2j+1}(2-2m)$ and $C_{2j+1}(2m)$ could be used, where $C_j(2m) = 2\cos(j\cos^{-1}m)$. A disadvantage in doing so is the non-vanishing of such polynomials at tabular points. To overcome this difficulty Kopal [1] and Miller [7] have suggested an expansion in terms of the polynomials

$$C_{2j+1}\left\{(2-2m)\cos\frac{\pi}{4j+2}\right\} \quad \text{and} \quad C_{2j+1}\left\{2m\cos\frac{\pi}{4j+2}\right\}.$$

Such an expansion Kopal calls the *Modified Chebyshev-Everett Interpolation Formula* and states that "$\cdots$ it represents potentially the most powerful and economic interpolation formula which can as yet be devised" [1]. See also Clenshaw and Olver [9].

Explicitly, following [1], p. 513, let

$$(1)\quad f(m) = f(0) + m\Delta_1 + \sum_{j=1}^{n}\left[\beta_0{}^jC_{2j+1}\left\{(2-2m)\cos\frac{\pi}{4j+2}\right\} + \beta_1{}^jC_{2j+1}\left\{2m\cos\frac{\pi}{4j+2}\right\}\right]$$

$$(2)\quad = f(0) + m\Delta_1 + \sum_{j=1}^{n}\frac{\sec^{2j+1}\left(\frac{\pi}{4j+2}\right)}{2^{2j+1}(2j+1)!} \times\left[N_0^{(2j)}C_{2j+1}\left\{(2-2m)\cos\frac{\pi}{4j+2}\right\} + N_1^{(2j)}C_{2j+1}\left\{2m\cos\frac{\pi}{4j+2}\right\}\right]$$

where the coefficients $N_0^{(2j)}$ and $N_1^{(2j)}$ will be designated *modified differences* of the $2j$-th order. Owing to the symmetry of the Everett coefficients $E^{(2n)}$ and to the

fact that $C_{2j+1}(x)$ is an odd function of x, the $N_1^{(2j)}$'s will be the same function of the Δ_1's as the $N_0^{(2j)}$'s are of the Δ_0's.

2. Calculation of the Modified Differences $N_{0,1}^{(2j)}$. The coefficients $N_{0,1}^{(2j)}$ are determined from the equivalence of the two expressions

$$f(m) = f(0) + m\Delta_1 + \sum_{j=1}^{n} (E_0^{(2j)}\Delta_0^{(2j)} + E_1^{(2j)}\Delta_1^{(2j)})$$

$$= f(0) + m\Delta_1 + \sum_{j=1}^{n} \frac{\sec^{2j+1}\left(\frac{\pi}{4j+2}\right)}{2^{2j+1}(2j+1)!}$$

$$\times\left[N_0^{(2j)}C_{2j+1}\left\{(2-2m)\cos\frac{\pi}{4j+2}\right\} + N_1^{(2j)}C_{2j+1}\left\{2m\cos\frac{\pi}{4j+2}\right\}\right]$$

or, in effect, from

$$(3) \qquad \sum_{j=1}^{n} E_1^{(2j)}\Delta_1^{(2j)} = \sum_{j=1}^{n} \frac{\sec^{2j+1}\left(\frac{\pi}{4j+2}\right)}{2^{2j+1}(2j+1)!} N_1^{(2j)}C_{2j+1}\left\{2m\cos\frac{\pi}{4j+2}\right\}.$$

Both sides of (3) are expanded in powers of m and the coefficients of equal powers equated. The procedure is quite straightforward, but the algebra is rather unwieldy, and is not reproduced here.

The following general relations can be obtained by equating the coefficients of m^{2n-3} and m:

$$N^{(2n-4)} = \Delta^{(2n-4)} - \frac{1}{4}\left(\frac{n}{3} - \frac{1}{2(n-1)}\sec^2\frac{\pi}{4n-2}\right)\Delta^{(2n-2)}$$

$$+\left\{\frac{(n+1)(5n+6)}{1440} - \frac{\sec^4\frac{\pi}{4n+2}}{64n(2n-1)} - \frac{\sec^2\frac{\pi}{4n-2}}{32(n-1)}\right.$$

$$\left.\times\left(\frac{n+1}{3} - \frac{1}{2n}\sec^2\frac{\pi}{4n+2}\right)\right\}\Delta^{(2n)} + \cdots$$

and

$$\sum_{j=1}^{n} (-1)^j \frac{(j!)^2}{(2j+1)!}\Delta_1^{(2j)} = \sum_{j=1}^{n} (-1)^j \frac{\sec^{2j}\frac{\pi}{4j+2}}{2^{2j}(2j)!} N_1^{(2j)}.$$

These relations are of assistance in computing the modified differences.

If differences up to the tenth are retained the following closed forms for the coefficients N^{ii}, N^{iv}, $\cdots$, N^{x} are obtained:

$$\begin{aligned}
N^{x} &= \Delta^{x}, \\
N^{viii} &= \Delta^{viii} + a_8\Delta^{x}, \\
N^{vi} &= \Delta^{vi} + a_4\Delta^{viii} + b_8\Delta^{x}, \\
N^{iv} &= \Delta^{iv} + a_2\Delta^{vi} + b_4\Delta^{viii} + c_8\Delta^{x}, \\
N^{ii} &= \Delta^{ii} + a_2\Delta^{iv} + b_2\Delta^{vi} + c_4\Delta^{viii} + d_8\Delta^{x},
\end{aligned}$$

where

$$a_j = -\frac{1}{4}\left(\frac{j+1}{3} - \frac{1}{2j}\sec^2\frac{\pi}{4j+2}\right),$$

$$b_j = \frac{(j+1)(5j+6)}{1440} - \frac{\sec^4\dfrac{\pi}{4j+2}}{64j(2j-1)} + a_j\frac{\sec^2\dfrac{\pi}{4j-2}}{8(j-1)},$$

$$c_8 = -\frac{139}{6048} + \frac{1}{5\cdot 3^8\cdot 2^{11}}\sec^6\frac{\pi}{22} - \frac{a_8}{7\cdot 2^8}\sec^4\frac{\pi}{18} + \frac{b_8}{24}\sec^2\frac{\pi}{14},$$

$$c_4 = -\frac{41}{3024} + \frac{5}{16\cdot 8!}\sec^6\frac{\pi}{18} - \frac{a_4}{960}\sec^4\frac{\pi}{14} + \frac{b_4}{16}\sec^2\frac{\pi}{10},$$

$$d_8 = \frac{479}{151200} - \frac{15}{2^7\cdot 10!}\sec^8\frac{\pi}{22} + \frac{5a_8}{16\cdot 8!}\sec^8\frac{\pi}{18} - \frac{b_8}{8\cdot 5!}\sec^4\frac{\pi}{14} + \frac{c_8}{16}\sec^2\frac{\pi}{10}.$$

The numerical values of the coefficients a, b, c, d-are given in Table I.

If the $N^{(2i)}$ are multiplied by $\dfrac{\sec^{2j+1}\left(\dfrac{\pi}{4j+2}\right)}{2^{2j+1}(2j+1)!}$, the actual coefficients of the Chebyshev polynomials $\beta_{0,1}{}^{(j)}$ in equation (1) are obtained. These are given in Table II ($i = 0,1$).

As will be seen, the coefficients $\beta^{(j)}$ diminish very rapidly and in practice it will only be necessary to use about two terms of the series (1). Should it be found unnecessary to retain the higher differences, they should simply be omitted from the $\beta^{(j)}$ or $N^{(2j)}$.

3. Besselian and Comrie-type Modified Differences. Clearly, the well-known Bessel interpolation formula can also be regrouped in terms of the Chebyshev polynomials, and the modified differences corresponding to it obtained in a similar manner. Modified differences arising in connection with Bessel's formula were first introduced into computational practice by E. W. Brown [3], and subsequently (but independently) by Camp [4] and Comrie [5]. Comrie, in particular, has done so much to propagate their use that they are usually associated with his name. It must be emphasized that the Comrie type of (Besselian) modified differences is not identical with what may be called the Chebyshev-Bessel modified differences, which would be obtained by an analogous procedure to that developed in the preceding sections of this note.

TABLE I

	Δ_t^{II}	Δ_t^{IV}	Δ_t^{VI}	Δ_t^{VIII}	Δ_t^{X}
N_t^{II}	1	−0.18090 16994 37	0.03717 66239 55	−0.00807 38992 79	0.00180 85304 04
N_t^{IV}		1	−0.28949 60381 83	0.07283 14707 77	−0.01772 12103 93
N_t^{VI}			1	−0.38444 50665 04	0.11351 63325 91
N_t^{VIII}				1	−0.47448 31950 54
N_t^{X}					1

TABLE II

	Δ^{II}	Δ^{IV}	Δ^{VI}	Δ^{VIII}	Δ^{X}	j	$\cos\frac{\pi}{4j+2}$
β_t^1	0.03207 50149 55	−0.00580 24247 15	0.00119 24407 69	−0.00025 89704 40	0.00005 80086 40	1	0.86602 54037 84
β_t^3		0.00033 46856 93	−0.00009 68901 81	0.00002 43756 51	−0.00000 59310 36	2	0.95105 65162 95
β_t^5			0.00000 18516 20	−0.00000 07118 46	0.00000 02101 89	3	0.97492 79121 82
β_t^7				0.00000 00061 77	−0.00000 00029 31	4	0.98480 77530 12
β_t^9					0.00000 00000 14	5	0.98982 14418 81

Comrie used modified differences $M^{(i)}$ whose coefficients were determined to minimize the error of the Bessel interpolation formula truncated after a given number of terms. Comrie never published details of the process by which these constants were determined; his argument has been reconstructed in [1], Chapter II. Although these constants were determined by Comrie with specific reference to Bessel's formula, it has become customary to use them in Everett's formula also. In this case an expansion

$$\begin{aligned} f(m) = f(0) + m\Delta_{\frac{1}{2}} + E_0^{\text{ii}}\Delta_0^{\text{ii}} + E_1^{\text{ii}}\Delta_1^{\text{ii}} + \cdots \\ + E_0^{(2j)}(\Delta_0^{(2j)} + A_j\Delta_0^{(2j+2)} + B_j\Delta_0^{(2j+4)} + \cdots) \\ + E_1^{(2j)}(\Delta_1^{(2j)} + A_j\Delta_1^{(2j+2)} + B_j\Delta_1^{(2j+4)} + \cdots) \end{aligned}$$

is used, where the coefficients A_j, B_j are those determined by Comrie for the Bessel formula.

For convenience, write

$$r_{2j+1} = \frac{\sec^{2j+1}\left(\dfrac{\pi}{4j+2}\right)}{2^{2j+1}(2j+1)!}\, C_{2j+1}\left(2m\cos\frac{\pi}{4j+2}\right).$$

Consider a specific example where differences up to the eighth are retained. Then $f(m)$ may be expanded:

$$\text{(4)}\quad f(m) = f(0) + m\Delta_{\frac{1}{2}} + E_1^{\text{ii}}\Delta_1^{\text{ii}} + E_1^{\text{iv}}\Delta_1^{\text{iv}} + E_1^{\text{vi}}\Delta_1^{\text{vi}} + E_1^{\text{viii}}\Delta_1^{\text{viii}} + \text{similar terms in } E_0\Delta_0,$$

$$\begin{aligned} \text{(5)}\quad f(m) = f(0) + m\Delta_{\frac{1}{2}} + r_3(\Delta_1^{\text{ii}} + a\Delta^{\text{iv}} + b\Delta_1^{\text{vi}} + c\Delta_1^{\text{viii}}) \\ + r_5(\Delta_1^{\text{iv}} + d\Delta_1^{\text{vi}} + e\Delta_1^{\text{viii}}) \\ + r_7(\Delta_1^{\text{vi}} + f\Delta_1^{\text{viii}}) \\ + r_9\Delta_1^{\text{viii}} + \cdots. \end{aligned}$$

A comparison of these formulae shows that

$$\begin{aligned} r_3 &= E_1^{\text{ii}} \\ ar_3 + r_5 &= E_1^{\text{iv}}. \end{aligned}$$

Now if we regard the last three terms of (5) as error terms then

$$\begin{aligned} f(m) &\sim f(0) + m\Delta_{\frac{1}{2}} + r_3(\Delta_1^{\text{ii}} + a\Delta_1^{\text{iv}} + b\Delta_1^{\text{vi}} + c\Delta_1^{\text{viii}}) + \cdots \\ &= f(0) + m\Delta_{\frac{1}{2}} + E_1^{\text{ii}}N_1^{\text{ii}} + \cdots. \end{aligned}$$

Reference to Comrie's paper [5] shows that N_1^{ii} is in fact identical with the modified second difference used by Comrie.

However, there seems to be no simple relation between N_1^{iv} and higher order modified differences, and the higher Comrie modified differences.

4. Example. As an example, an interpolation has been made in a table of Fermi-Dirac functions at present being prepared by the author.

A portion of this table, together with 1st, 2nd, 4th, 6th, 8th, and 10th differences, is reproduced below. It has been chosen because the differences do not fall off rapidly, as can be seen.

η	$F(\eta)$	$\Delta_{\frac{1}{2}}$	Δ^{ii}	Δ^{iv}	Δ^{vi}	Δ^{viii}	Δ^{x}
−2.0	0.1145 8783						
		6921 403					
−1.5	1838 0186		3748 501	297 363	−135 029	65 578	−20 496
		10669 904					
−1.0	2905 0090		5259 311	130 749	−125 114	110 942	−119 432
		15929 215					
−0.5	*4497 9305*		*6900 870*	*−160 979*	*−4 256*	*36 873*	*−68 774*
		22830 085					
0.0	6780 9389		*8381 450*	*−456 963*	*153 475*	*−105 970*	*113 458*
		31211 535					
0.5	9902 0925		9405 068	−599 471	205 237	−135 355	132 585
		40616 603					
1.0	1.3963 7528		9829 215	−536 743	121 643	−32 156	−17 467
		50445 818					
1.5	1.9008 3346		9716 618	−352 371	5 895	53 577	−82 922
		60162 437					
2.0	2.5024 5783						

From this table we find, e.g., the value of $F(-0.3)$ from the formula: $(m=0.4)$,

$$F(-0.3) = F(-0.5) + 0.4\Delta_{\frac{1}{2}} + E_0^{ii}(0.4)N_0^{ii} + E_1^{ii}(0.4)N_1^{ii} + \beta_0^2 C_5\left(1.2\cos\frac{\pi}{10}\right) + \beta_1^2 C_5\left(0.8\cos\frac{\pi}{10}\right)$$

where

$$N_0^{ii} = 0.06929\ 411 \qquad \beta_0^2 = -\ 0.00000\ 052$$
$$N_1^{ii} = 0.08470\ 882 \qquad \beta_1^2 = -\ 0.00000\ 171.$$

The values of $C_5\left(1.2\cos\frac{\pi}{10}\right)$ and $C_5\left(0.8\cos\frac{\pi}{10}\right)$ are obtained by linear interpolation in the table of Chebyshev polynomials [6].

For comparison, columns (1) and (2) below give respectively the value of $F(-0.3)$, as calculated with the same number of terms in the interpolation formula, by the straight Everett formula,

$$f(m) = f(0) + m\Delta_{\frac{1}{2}} + E_0^{ii}\Delta_0^{ii} + E_1^{ii}\Delta_1^{ii} + E_0^{iv}\Delta_0^{iv} + E_1^{iv}\Delta_1^{iv}$$

and the modified Everett formula using Comrie's coefficients in the modified fourth difference,

$$f(m) = f(0) + m\Delta_{\frac{1}{2}} + E_0^{ii}\Delta_0^{ii} + E_1^{ii}\Delta_1^{ii} + E_0^{iv}M_0^{iv} + E_1^{iv}M_1^{iv}$$

where $M_{0,1}^{iv} = \Delta_{0,1}^{iv} - 0.207\Delta_{0,1}^{vi} + 0.045\Delta_{0,1}^{viii}$. Column (3) gives the value computed by the present method. The value obtained by direct calculation is 0.53193 157. It will be seen that the present method gives substantially better results for this case.

		1.	2.	3.
$F(-0.3) =$		0.44979305	0.44979305	0.44979305
	+	9132034	9132034	9132034
	−	441656	441656	443482
	−	469361	469361	474369
	−	1875	1845	11
	−	4913	5306	318
		0.53193534	0.53193171	0.53193159

In most applications high order differences will fall off rapidly and probably only the modified second difference will be required. If this is calculated and printed with a table, interpolation becomes a very simple procedure.

5. Use of the Expansion. An interesting discussion about the merits of a Chebyshev-Everett interpolation formula is recorded in Appendix 11 of reference [7] to which the reader is referred. (This report was received after the major part of this note had been written.)

It should be emphasized that the chief advantage of this type of expansion is that the number of terms in the interpolating expansion of a function $f(m)$, dependent on m, is reduced considerably. Clearly, the expansion will have its greatest use when several interpolations within a given interval are required. The complicated modified differences, once calculated, are available for all interpolations within the relevant interval, the number of terms dependent on m being usually only two, *viz.*,

$$C_s\left\{2m\cos\frac{\pi}{6}\right\} \quad \text{and} \quad C_s\left\{(2-2m)\cos\frac{\pi}{6}\right\}.$$

A table of the above functions would be advantageous, especially when the expansion is being used for sub-tabulation. For purposes other than sub-tabulation, values of the Chebyshev functions can be obtained by simple interpolation in the existing tables [6] and [8]. For convenience, values of $\cos\dfrac{\pi}{4j+2}$ are given to twelve decimal places in Table II.

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Note added in proof. Dr. L. Fox of the National Physical Laboratory, Teddington, Middlesex, England, has recently informed the author in a paper as yet unpublished, that in fact the Everett form of the modified Chebyshev-Bessel interpolation formula gives slightly better results (in the sense that the maximum possible residual error is smaller) than the formula advocated here.

Iterative Procedure for Evaluating a Transient ResponseThrough its Power Series

Introduction. We discuss here a particular method of evaluating a time function, such as a transient response, from its Laplace transform. We shall assume the Laplace transform of $F(t)$,

$$\bar{F}(S) = \int_0^\infty e^{-st} F(t)\, dt \tag{1}$$

is given, and is of the form

$$\bar{F}(S) = \frac{A_0^0 S^{N-1} + A_1^0 S^{N-2} + \cdots + A^0_{N-2} S + A^0_{N-1}}{S^N + b_1 S^{N-1} + b_2 S^{N-2} + \cdots + b_{N-1} S + b_N}. \tag{2}$$

(The superscript 0 of the A's does not indicate a power, but is used as a superscript for reasons which will appear presently.)

We require in (2) that the denominator be of higher degree than the numerator, which is necessary if $F(t)$ is a Laplace transform. We shall always assume also that we have $\bar{F}(S)$ written with the coefficient of S^N in the denominator equal to unity, as in equation (2).

The method described below is an iterative method for obtaining the quantities α_j in MacLaurin series for $F(t)$, which we write in the form

$$F(t) = \sum_{j=0}^{\infty} \alpha_j \frac{t^j}{j!}. \tag{3}$$

The method has the advantage that it is not necessary to know the roots of the numerator or denominator of $\bar{F}(S)$. The series (3) will converge for all positive values of t, although for larger values of t a great many terms may be needed. This, however, is not too great a disadvantage for digital purposes, since the method for computing the α_j's is an iterative one. It is necessary, however, to have some idea of how many terms will be needed, and a criterion for this is also given below and discussed. The theoretical derivations are given in the Appendix.

Method for Computing the α_j's. The method for computing the α_j's, the theory of which is discussed in the Appendix, is as follows: We start with the coefficients

$$A_0^0, A_1^0, \cdots, A^0_{N-1}, b_1, b_2, \cdots, b_N \quad \text{of (2).}$$

Then

$$\alpha_0 = A_0^0.$$

We next evaluate the numbers $A_0^1, A_1^1, \cdots, A^1_{N-1}$ as follows:

$$\begin{aligned} A_0^1 &= A_1^0 - \alpha_0 b_1 \\ A_1^1 &= A_2^0 - \alpha_0 b_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ A^1_{N-2} &= A^0_{N-1} - \alpha_0 b_{N-1} \\ A^1_{N-1} &= -\alpha_0 b_N. \end{aligned} \tag{4}$$

Then

$$\alpha_1 = A_0^1.$$

Then we evaluate

$$\begin{aligned} A_0^2 &= A_1^1 - \alpha_1 b_1 \\ A_1^2 &= A_2^1 - \alpha_1 b_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ A^2_{N-2} &= A^1_{N-1} - \alpha_1 b_{N-1} \\ A^2_{N-1} &= -\alpha_1 b_N \end{aligned} \tag{5}$$

and $\alpha_2 = A_0^2$, and so on. (Again the superscripts 1, 2, etc., are merely superscripts and do not indicate powers of the A's.) In contracted notation we have

$$\begin{aligned} \alpha_j &= A_0^j \\ A_k^{j+1} &= A_{k+1}^j - \alpha_j b_{k+1}, \quad 0 \le k \le N-2 \\ A_{N-1}^{j+1} &= -\alpha_j b_N \end{aligned} \tag{6}$$

when $j = 0, 1, 2, 3$, etc.

Thus the numbers α_j can be obtained and substituted in equation (3) to give $F(t)$.

Number of Terms. It is difficult to estimate the exact number of terms of the series (3) necessary to give a prescribed accuracy. However, it is shown in the Appendix that if we wish to have $F(t)$ accurate to within $\pm\epsilon$ with $0 < t \le t_0$, neglecting numerical round-off errors, it is *sufficient* to have $n > n_0$, where $(n+1)$ is the number of terms (counting the 0 order term), and

$$n_0 = e(1+b)t_0 + \ln\left[\frac{A(1+b)[(1+b)^N - 1]}{2b\epsilon(e-1)}\right]; \tag{7}$$

here A is the maximum of the numbers $|A_0^0|, |A_1^0|, \cdots, |A^0_{N-1}|$; b is the maximum of the numbers $|b_1|, |b_2|, \cdots, |b_N|$; and e is the base of the natural logarithms—"ln" denotes the natural logarithm.

It should be borne in mind that the above criterion may give more terms than are actually necessary in a particular case. However, it has the advantage that it offers a method of determining the number of terms from the function $\bar{F}(S)$ alone, without necessarily having advance knowledge concerning the function $F(t)$.

As an illustration we consider two simple cases where $F(t)$ is easily evaluated by elementary methods. First consider

$$\bar{F}(S) = \frac{1}{S+1}.$$

Then we know

$$F(t) = e^{-t}.$$

(Our iterative procedure will give $\alpha_j = (-1)^j$.)

In this case, $A = b = N = 1$. If we take $t_0 = 5$ seconds, and $\epsilon = 10^{-8}$, substitution in equation (7) gives $n_0 = 31.3$, so we would use 33 terms (counting the 0 order term). However if we take $n = 17$, our error is less than $5^{18}/18!$ which is less than 10^{-8}. $5^{17}/17!$ is greater than 10^{-8}. Thus the actual number that we need in this case is $n = 17$, or 18 terms, so that equation (7) gave us far too many.

Next consider

$$\bar{F}(S) = \frac{2S+11}{S^2+11S+10} \quad (N = 2, A = b = 11).$$

Here again we can find directly that

$$F(t) = e^{-t} + e^{-10t}$$

so in this case

$$\alpha_j = (-1)^j[10^j + 1].$$

Substitution in (7), with the same t_0 and ϵ as in the previous case, gives $n_0 = 171.6$ so that we would use 173 terms according to our formula. The number of terms that is actually needed in this case is around 145. Thus in this case again our formula gives too many terms, but not by as large a ratio as in the previous case.

Appendix

The method under discussion is based on the expansion of $\bar{F}(S)$ around the point at infinity. Thus we substitute $1/Z$ for S and get

$$f(Z) = \bar{F}\left(\frac{1}{Z}\right) = \frac{Z\{A_0^0 + A_1^0 Z + \cdots + A^0_{N-1}Z^{N-1}\}}{1 + b_1 Z + \cdots + b_N Z^N}. \tag{8}$$

Next we write

$$f(Z) = Z_g(Z) \tag{9}$$

so

$$g(Z) = \frac{p_0(Z)}{Q(Z)} \tag{10}$$

where

$$p_0(Z) = A_0^0 + A_1^0 Z + \cdots + A^0{}_{N-1} Z^{N-1} \tag{11}$$

and

$$Q(Z) = 1 + b_1 Z + \cdots + b_N Z^N. \tag{12}$$

Now $g(Z)$ is analytic at the origin ($Z = 0$), and so it has a MacLaurin series:

$$g(Z) = \alpha_0 + \alpha_1 Z + \alpha_2 Z^2 + \cdots. \tag{13}$$

We shall show presently that these coefficients α_j, are the same as the α_j in equation (3), and also that the series (3) converges for all values of t.

To evaluate the α_j in equation (13) we use the method of [1], which becomes somewhat simpler in this case since we are expanding around the origin. From (10) and (13) we have

$$p_0(Z) = Q(Z)[\alpha_0 + \alpha_1 Z + \alpha_2 Z^2 + \cdots]. \tag{14}$$

Then

$$\alpha_0 = \frac{p_0(0)}{Q(0)} = A_0^0. \tag{15}$$

Now, having α_0, we note that

$$p_0(Z) - \alpha_0 Q(Z)$$

has 0 as a root, so we can define another polynomial

$$p_1(Z) = \frac{p_0(Z) - \alpha_0 Q(Z)}{Z}. \tag{16}$$

Then

$$p_1(Z) = A_0^1 + A_1^1 Z + A_2^1 Z^2 + \cdots + A^1{}_{N-1} Z^{N-1} \tag{17}$$

and the coefficients A_k^1 are given by equations (4). We then have

$$p_1(Z) = Q(Z)[\alpha_1 + \alpha_2 Z + \alpha_3 Z^2 + \cdots]. \tag{18}$$

Thus we get

$$\alpha_1 = \frac{p_1(0)}{Q(0)} = A_0^1. \tag{19}$$

Then defining the polynomial

$$p_2(Z) = \frac{p_1(Z) - \alpha_1 Q(Z)}{Z},$$

we get

$$p_2(Z) = A_0^2 + A_1^2 Z + \cdots + A^2_{N-1} Z^{N-1} \tag{20}$$

where the coefficients A_1^2 are given by equations (5). Then $\alpha_2 = A_0^2$. If we continue in this manner we get the iterative relationships (6).

Next we shall use the method of [2] to prove equation (3), since this will assist us in obtaining an error criterion. Suppose ϵ is a small positive number. We want to show that in a given interval, $0 - t_0$, we can choose n_0 such that when $n > n_0$,

$$\left| F(t) - \sum_{j=0}^{n} \alpha_j \frac{t^j}{j!} \right| < \epsilon, \quad 0 < t_0 \leq t_0.$$

First let r_1 be a positive number such that all the roots of $Q(Z)$ are greater than r_1 in absolute value. Then $g(Z)$ is analytic for $|Z| \leq r_1$, and so also is

$$h(Z) = \frac{p_1(Z)}{Q(Z)}. \tag{21}$$

From equation (18) we see that

$$h(Z) = \sum_{j=1}^{\infty} \alpha_j Z^{j-1}. \tag{22}$$

The series (22) will converge uniformly for $|Z| \leq r_1$. Thus if we let

$$R_n(Z) = \sum_{j=n+1}^{\infty} \alpha_j Z^{j-1} \tag{23}$$

then for any positive number ϵ_1 we can find $n_0(\epsilon_1)$ such that $|R_n(Z)| < \epsilon_1$ when $|Z| \leq r_1$ and $n > n_0(\epsilon_1)$.

ϵ_1 can be any positive number and we will choose

$$\epsilon_1 = \frac{2\epsilon}{r_1} e^{-\frac{t_0}{r_1}}. \tag{24}$$

Since the point $\frac{1}{r_1} + i0$ is to the right of all the singularities of $\bar{F}(S)$, we have by the inversion theorem

$$2\pi i F(t) = \int_{\frac{1}{r_1} - i\infty}^{\frac{1}{r_1} + i\infty} \bar{F}(S) e^{St} dS = \int_{\frac{1}{r_1} - i\infty}^{\frac{1}{r_1} + i\infty} e^{St} \left\{ \sum_{j=0}^{\infty} \frac{\alpha_j}{S^{j+1}} \right\} dS. \tag{25}$$

The last step follows from equations (8), (9), and (13) substituting $1/S$ for Z.

Next we consider the function

$$F_n(t) = \sum_{j=0}^{n} \alpha_j \frac{t^j}{j!}. \tag{26}$$

The Laplace transform of $t^j/j!$ is $\frac{1}{S^{j+1}}$, $(j = 0, 1, 2, \cdots)$.

Thus by the inversion theorem

$$2\pi i F_n(t) = \sum_{j=0}^{n} \alpha_j \left[\int_{\frac{1}{r_1}-i\infty}^{\frac{1}{r_1}+i\infty} \frac{e^{St}}{S^{j+1}}\, dS \right]. \tag{27}$$

Then for $n > n_0(\epsilon_1)$, we have

$$\begin{aligned} &|F(t) - F_n(t)| \\ &= \frac{1}{2\pi}\left| \int_{\frac{1}{r_1}-i\infty}^{\frac{1}{r_1}+i\infty} e^{St} \sum_{j=0}^{\infty} \frac{\alpha_j}{S^{j+1}}\, dS - \sum_{j=0}^{n} \alpha_j \int_{\frac{1}{r_1}-i\infty}^{\frac{1}{r_1}+i\infty} \frac{e^{St}}{S^{j+1}}\, dS \right| \\ &= \frac{1}{2\pi}\left| \int_{\frac{1}{r_1}-i\infty}^{\frac{1}{r_1}+i\infty} e^{St} \frac{R_n\left(\frac{1}{S}\right)}{S^2}\, dS \right| = \left| \int_{-\infty}^{\infty} \left\{ \frac{R_n\left(\frac{1}{\frac{1}{r_1}+iy}\right) i}{\left(\frac{1}{r_1}+iy\right)^2} \right\} dy \right| \cdot \frac{1}{2\pi} \\ &< \frac{\epsilon_1 e^{\frac{t}{r_1}}}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{\frac{1}{r_1^2}+y^2} = \frac{\epsilon e^{\frac{t-t_0}{r_1}}}{r_1 \pi} [r_1 \pi] \leq \epsilon \end{aligned} \tag{28}$$

if $0 < t \leq t_0$. The numbers, t_0 and ϵ may have any positive values. Once they are chosen we choose ϵ_1 by equation (24), and then find $n_0(\epsilon_1)$. The actual finding of a number n_0, after having found ϵ_1 is the task which we will consider next. However we have proved that the series of equation (3) converges to $F(t)$ for all positive t. Also we see that the number n_0 which makes $|F(t) - F_n(t)| < \epsilon$ for $t = t_0$, also does so for all values of t between 0 and t_0.

Next we ask the question: Having given ϵ_1 from equation (24), how do we actually find a number n_0 such that $|R_n(Z)| < \epsilon_1$ when $|Z| \leq r_1$ and $n > n_0$? We must do this in order to have a formula for obtaining a sufficient number of terms in the series (3). Also we have not yet chosen r_1, except to say that it is less than the modulus of each of the roots of $Q(Z)$.

First of all we define

$$r_0 = \frac{1}{1+b} \tag{29}$$

where b is the maximum of the numbers $|b_1|, |b_2|, \cdots, |b_N|$. Then for $|Z| \leq r_0$,

$$\begin{aligned} |Q(Z)| &\geq 1 - b\left[\frac{1}{1+b} + \left(\frac{1}{1+b}\right)^2 + \cdots + \left(\frac{1}{1+b}\right)^N\right] \\ &= 1 - \left(\frac{b}{1+b}\right) \frac{1 - \left(\frac{1}{1+b}\right)^N}{1 - \left(\frac{1}{1+b}\right)} = \frac{1}{(1+b)^N}. \end{aligned} \tag{30}$$

Thus all the roots of $Q(Z)$ are outside the circle $|Z| = r_0$. Now we choose

$$r_1 = \theta r_0 = \frac{\theta}{1+b}, \tag{31}$$

where θ is as yet undetermined, except that

$$0 < \theta < 1.$$

Then for $|Z| \leq r_1$, we have

$$R_n(Z) = \sum_{j=n+1}^{\infty} a_j Z^{j-1} = h(Z) - \sum_{j=1}^{n} a_j Z^{j-1}. \tag{32}$$

Also

$$a_j = \frac{h^{(j-1)}(0)}{(j-1)!} \tag{33}$$

where $h^{(k)}(Z)$ is the k^{th} derivative of $h(Z)$. From Cauchy's integral formula, we have

$$h(Z) = \frac{1}{2\pi i} \oint_{C_0} \frac{h(w)}{w - Z}\, dw \tag{34}$$

and

$$h^{(k)}(Z) = \frac{k!}{2\pi i} \oint_{C_0} \frac{h(w)}{(w-Z)^{k+1}}\, dw \tag{35}$$

if $|Z| \leq r_1$, where C_0 is the circle $|w| = r_0$.

Thus, for $|Z| \leq r_1$,

$$R_n(Z) = \frac{1}{2\pi i} \oint_{C_0} \frac{h(w)}{w - Z}\, dw - \sum_{j=1}^{n} \frac{Z^{j-1}}{2\pi i} \oint_{C_0} \frac{h(w)}{w^j}\, dw. \tag{36}$$

The second term

$$\sum_{j=1}^{n} \frac{Z^{j-1}}{2\pi i} \oint_{C_0} \frac{h(w)}{w^j}\, dw \tag{37}$$

$$= \frac{1}{2\pi i} \oint_{C_0} \frac{h(w)}{w} \sum_{j=1}^{n} \left(\frac{Z}{w}\right)^{j-1} dw = \frac{1}{2\pi i} \oint_{C_0} \frac{h(w)}{w} \frac{\left(1 - \frac{Z}{w}\right)^n}{\left(1 - \frac{Z}{w}\right)}\, dw$$

$$= \frac{1}{2\pi i} \oint_{C_0} \frac{h(w)}{w - Z}\, dw - \frac{1}{2\pi i} \oint_{C_0} \frac{h(w)}{(w-Z)} \left(\frac{Z}{w}\right)^n dw.$$

Hence

$$R_n(Z) = \frac{1}{2\pi i} \oint_{C_0} \frac{h(w)}{(w-Z)} \left(\frac{Z}{w}\right)^n dw. \tag{38}$$

Thus, since $|Z| \leq r_1 = \theta r_0$,

$$|R_n(Z)| \leq \frac{1}{2\pi} \frac{M}{r_0(1-\theta)} \theta^n 2\pi r_0 = \frac{M\theta^n}{(1-\theta)} \tag{39}$$

where M is the maximum of the function $|h(w)|$ on the circle C_0.

On the circle C_0, we have

$$|h(w)| = \frac{|p_1(w)|}{|Q(w)|} \leq (1+b)^N |p_1(w)|, \tag{40}$$

using (30).

The coefficients A_k^1 of $p_1(Z)$ are given by equations (4), and since $\alpha_0 = A_0^0$, we see from these equations that

$$|A_k^1| \leq A(1+b), \quad k = 0, 1, \cdots, (N-1), \tag{41}$$

where A is the maximum of the numbers $|A_0^0|, |A_1^0|, \cdots, |A_{N-1}^0|$.

Thus on the circle C_0,

$$\begin{aligned} |p_1(w)| &\leq A(1+b)\left[1 + \frac{1}{1+b} + \left(\frac{1}{1+b}\right)^2 + \cdots + \left(\frac{1}{1+b}\right)^{N-1}\right] \\ &= A(1+b) \frac{1 - \left(\frac{1}{1+b}\right)^N}{1 - \frac{1}{1+b}}. \end{aligned} \tag{42}$$

Therefore, using (40),

$$M \leq A(1+b)^{N+1} \frac{1 - \left(\frac{1}{1+b}\right)^N}{1 - \frac{1}{1+b}} = \frac{A(1+b)^2[(1+b)^N - 1]}{b}.$$

Thus, returning to condition (39) we see that for $|Z| \leq r_1$,

$$|R_n(Z)| \leq \frac{A(1+b)^2[(1+b)^N - 1]\theta^n}{b(1-\theta)}. \tag{43}$$

Thus, we wish to have

$$\frac{A(1+b)^2[(1+b)^N - 1]\theta^n}{b(1-\theta)} < \epsilon_1.$$

This will be true if

$$\left(\frac{1}{\theta}\right)^n > \frac{A(1+b)^2[(1+b)^N - 1]}{b\epsilon_1(1-\theta)},$$

that is,

$$n > \frac{\ln\left[\frac{A(1+b)^2[(1+b)^N - 1]}{b\epsilon_1(1-\theta)}\right]}{\ln\left(\frac{1}{\theta}\right)}.$$

Thus we have found a number $n_0(\epsilon_1)$ which has the required property, namely

$$n_0 = \frac{\ln\left[\dfrac{A(1+b)^2[(1+b)^N - 1]}{b\epsilon_1(1-\theta)}\right]}{\ln\left(\dfrac{1}{\theta}\right)}. \tag{44}$$

In order to relate ϵ_1 to ϵ, the desired maximum error in $F(t)$, and the interval (0 to t_0), we use equations (24) and (31), and get

$$\begin{aligned} n_0 &= \frac{\ln\left[\dfrac{A(1+b)[(1+b)^N - 1]\theta e^{\frac{(1+b)t_0}{\theta}}}{2b(1-\theta)\epsilon}\right]}{\ln\left(\dfrac{1}{\theta}\right)} \\ &= \frac{(1+b)t_0}{\theta \ln\left(\dfrac{1}{\theta}\right)} + \frac{\ln\left[\dfrac{A(1+b)[(1+b)^N - 1]\theta}{2b(1-\theta)\epsilon}\right]}{\ln\left(\dfrac{1}{\theta}\right)}. \end{aligned} \tag{45}$$

We still have not chosen θ, except to say that it is a positive number less than unity. We somewhat arbitrarily choose θ so as to make the t_0 term as small as possible, i.e., the value which makes $\theta \ln (1/\theta)$ maximum. This value is

$$\theta = e^{-1}. \tag{46}$$

Thus

$$n_0 = e(1+b)t_0 + \ln\left[\frac{A(1+b)[(1+b)^N - 1]}{2b\epsilon(e-1)}\right] \tag{47}$$

which is the same as equation (7).

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The Numerical Integration of Two-Point Boundary Value Problems

1. Introduction. A general system of linear, ordinary, differential equations may be written, in matrix notation

$$\dot{Y} = A(t)Y + F(t), \tag{1}$$

where the dot denotes differentiation with respect to the independent variable, t; Y is the column matrix of unknowns,

$$\begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix},$$

$A(t)$ is a square matrix, and $F(t)$, the forcing functions, is another column matrix,

$$\begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}.$$

Suppose that r values $y_1(0) \cdots y_r(0)$ are given and at $t = T$, $(n - r)$ of the y_i's are given, *viz.*: y_{i_k}, $k = 1, \cdots, (n - r)$. In order to solve (1) using high speed computing machinery, whether analogue or digital, it is essential to have n conditions at one end of the interval $0 - T$. One possibility is to carry the constants $y_{r+1}(0) \cdots y_n(0)$ along as unknowns during the integration, and then to determine them when the far end is reached. Another possibility consists of guessing $y_{r+1}(0) \cdots y_n(0)$ and then adjusting them. These techniques have been described by Tifford [1]. The first of these methods is awkward especially if a large number of unknowns must be carried; the second requires more integrations than the methods described below.

This paper is concerned with methods for converting a two-point boundary value problem into an initial value problem. For linear systems the conversion is accomplished with a finite number of integrations. In section 2, where nonlinear systems are considered, an iterative procedure is required.

2. Linear Systems—Method of Adjoint Equations. The adjoint system was developed by Bliss [2]; more recent developments are described by Wright [3]. The system adjoint to (1) is, by definition,

$$-\dot{X} = \bar{A}(t)X, \tag{2}$$

where the bar denotes the transposed matrix and X is a column matrix

$$\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}.$$

The x_i's and y_i's are related by

$$\frac{d}{dt}(x_i y_i) = x_i f_i, \tag{3}$$

where the repeated suffix implies summation from 1 to n. A proof of (3) follows: Consider

$$\frac{d}{dt}(x_i y_i) \equiv \frac{d}{dt}(\bar{X}Y) = \bar{X}\dot{Y} + \dot{\bar{X}}Y.$$

Substituting for the derivatives from (1) and (2) there is obtained

$$\begin{aligned} \frac{d}{dt}(x_i y_i) &= \bar{X}(AY + F) - (\overline{\bar{A}X})Y \\ &= \bar{X}AY + \bar{X}F - (\bar{X}A)Y. \end{aligned}$$

The first and last terms cancel identically and the result (3) follows.

Integrating (3) over the interval $[0,T]$, it is found that

$$x_i(T)y_i(T) - x_i(0)y_i(0) = \int_0^T x_i(t)f_i(t)dt. \tag{4}$$

This fundamental result is the one-dimensional form of Green's theorem.

The aim is now to calculate $y_{r+1}(0) \cdots y_n(0)$ using the given conditions. The method described in this section will utilize the adjoint system (2), whereas in the next section the original equations (1), only, will be used.

The adjoint system (2) must be integrated $(n - r)$ times from T to 0 subject to certain starting values. These are chosen such that the $x_i(T)$'s are all zero except one. For the m^{th} integration this exception is the x_i which appears as the coefficient of $y_{i_m}(T)$ in (4), and which may be designated by $x_{i_m}(T)$. For simplicity $x_{i_m}(T)$ is chosen to be unity.

Let ${}_m x_i(t)$ be the value of x_i deduced from the m^{th} integration of the adjoint system. The fundamental result (4) holds for each integration of the adjoint system, and this is expressed by rewriting it

$${}_m x_i(T)y_i(T) - {}_m x_i(0)y_i(0) = \int_0^T {}_m x_i(t)f_i(t)dt. \tag{5}$$

But since all $x_i(T)$'s were chosen zero except $x_{i_m}(T)$, the first summation reduces to a single term and (5) becomes

$${}_m x_i(0)y_i(0) - y_{i_m}(T) = \int_T^0 {}_m x_i(t)f_i(t)dt. \tag{6}$$

This is a system of $(n - r)$ linear algebraic equations for the $(n - r)$ unknowns $y_i(0)$, $i = r + 1 \cdots n$.

With these values determined (1) can now be integrated from 0 to T, as an initial value problem. The total number of integrations consists of $(n - r)$ of the adjoint system plus one of the original given system.

3. Linear Systems—Method of Complementary Functions. In this case the homogeneous equation, derived from (1),

$$\dot{u} = A(t)u \tag{7}$$

is integrated from 0 to T $(n - r)$ times. The m^{th} time the initial conditions are

$$\begin{aligned} {}_m u_i(0) &= 0 \quad i \neq r + m \\ &= 1 \quad i = r + m, \end{aligned}$$

where ${}_m u_i(t)$ is the result of the m^{th} integration of $u_i(t)$.

Next the non-homogeneous equation

$$\dot{V} = A(t)V + F(t) \tag{8}$$

is integrated once, from 0 to T, subject to

$$\begin{aligned} v_i(0) &= y_i(0) \quad i = 1 \cdots r \\ &= 0 \qquad i = r + 1 \cdots n. \end{aligned}$$

The general solution of (1), subject to the given conditions can be written

$$y_i(t) = C_p \, {}_p u_i(t) + v_i(t), \tag{9}$$

where the C_p's are constants determined from the $(n - r)$ equations

$$y_{i_k}(T) = C_p \, {}_p u_{i_k}(T) + v_{i_k}(T) \quad k = 1 \cdots (n - r). \tag{10}$$

In this equation the left-hand side is given and the quantities $u_i(T)$ and $v_i(T)$ have been determined from the integrations of (7) and (8). By letting $t = 0$ in (9) it is seen that $C_p = y_{r+p}(0)$, $p = 1 \cdots n - r$.

The method of complementary functions takes less time than the method of adjoint equations for a single problem, but it is not so suitable when solutions are required for a large number of different forcing functions, F, or for a large number of different boundary conditions.

4. Nonlinear Systems. A general system of nonlinear, ordinary, differential equations may be written

$$\dot{y}_i(t) = g_i(y_1 \cdots y_n, t) \quad i = 1 \cdots n, \tag{11}$$

where it is assumed that the functions g_i are once differentiable with respect to all the y_i's. Once again the boundary conditions are specified as $y_1(0) \cdots y_r(0)$ and $y_{i_k}(T)$, $k = 1 \cdots (n-r)$. The procedure is to estimate $y_{r+1}(0) \cdots y_n(0)$ which estimates are denoted by $y^*_{r+1}(0) \cdots y_n^*(0)$. Equations (11) can now be integrated as an initial value problem, and the solutions are denoted by $y_i^*(t)$. In general, it will be found that the computed values of $y_{i_k}(T)$, $k = 1 \cdots (n-r)$ namely, $y^*_{i_k}(T)$, differ from the given values of $y_{i_k}(T)$. To obtain the correct solution of the system (11) the difference between $y^*_{i_k}(T)$ and the given $y_{i_k}(T)$ must be made as small as possible.

Define $\delta y_i(t)$ by

$$\delta y_i(t) = y_i(t) - y_i^*(t) \quad i = 1 \cdots n. \tag{12}$$

Substituting $y_i(t)$ in (11) there results, to a first approximation,

$$\delta \dot{y}_i(t) = \left(\frac{\partial g_i}{\partial y_j}\right) \delta y_j. \tag{13}$$

The equations adjoint to (13) are

$$-\dot{x}_i = \left(\frac{\partial g_j}{\partial y_i}\right) x_j. \tag{14}$$

Since non-homogeneous terms are absent from (13) the Green's theorem for (13) and (14) reduces to

$$x_i(T)\delta y_i(T) - x_i(0)\delta y_i(0) = 0. \tag{15}$$

The quantities required are $\delta y_i(0)$, $i = r+1 \cdots n$. To obtain $(n-r)$ equations, for these $(n-r)$ unknowns, (14) is integrated from T to 0 $(n-r)$ times; the result of the m^{th} integration is denoted by ${}_m x_i(t)$. Equation (15) holds for each integration of the adjoint system, and this may be expressed by rewriting it

$${}_m x_i(T)\delta y_i(T) - {}_m x_i(0)\delta y_i(0) = 0 \quad m = 1 \cdots (n-r). \tag{16}$$

The m^{th} time the adjoint equations are integrated the starting values are that all $x_i(T)$'s are zero except the coefficient of $\delta y_{i_m}(T)$ which is chosen to be unity. The first summation in (16) now reduces to a single term; furthermore, $\delta y_i(0)$, $i = 1 \cdots r$ are identically zero. Hence (16) reduces to

$$\delta y_{i_m}(T) - {}_m x_{r+i}(0)\delta y_{r+i}(0) = 0 \quad m = 1 \cdots (n-r), \tag{17}$$

where the summation on i extends from 1 to $(n-r)$. Substituting these values into (12) the improved estimate of $y_i(0)$, $i = r+1 \cdots n$ are obtained. In the case of nonlinear systems (11) is integrated again, using the improved estimates, and the process is repeated. In the special case where the g_i's are linear functions

of y_i (13) is exact and the new estimates of the initial values are precise after only one application of the method. This method, when applied to linear systems, is similar to the second method described by Tifford [1].

5. Remarks. The problem was posed such that the y_i's were given at either end of the interval $0 - T$. The methods are easily extended, if instead, $n - p$ linear relations are prescribed between the $y_i(0)$'s and p between the $y_i(T)$'s. The methods may also be generalized to convert three or more point boundary value problems into an initial value problem.

An alternative method, suitable for nonlinear systems, has been presented by Lance and Deland [4]. This method is also one of iteration but it involves numerical differentiation and for this reason is less accurate per iteration than the method presented in section 4.

If large scale computing machines are not available then relaxation methods can be used to solve systems of differential equations. Allen [5] states that if relaxation methods are to be used then an equal number of boundary conditions must be specified at each end of the interval, and for this reason the number of equations, n, must be even. Allen presents a method for converting problems which are not stated in this form into this form. The authors considered using the methods given above as an alternative to Allen's. Although, it is possible to do this, Reid [6] has shown that the conversion is unnecessary and that relaxation can be used directly regardless of how the boundary conditions are distributed. Reid also shows that the number of equations need not be even.

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High Accuracy Quadrature Formulas from Divided Differences with Repeated Arguments

In the numerical solution of differential equations, one uses quadrature formulas to relate the unknown functions to their derivatives. While the functions enter only at the beginning and end of the range of integration, all intermediate values of the derivatives may enter. For checking purposes at least, it would be advantageous to increase the accuracy of the quadrature formulas by permitting weights to be assigned to intermediate values of the functions, as well as their derivatives. An interesting method of obtaining such formulas is given below.

Given a function $f(x)$ and $n+1$ distinct values of the argument $x_0, x_1, \cdots, x_n$, the nth order divided difference of $f(x)$ with respect to these values of x may be defined as

$$f(x_0, x_1, \cdots, x_n) = \sum_{p=0}^{n} \frac{f(x_p)}{\Pi_p}, \tag{1}$$

where

$$\Pi_p = \prod_{\substack{i=0 \\ i \neq p}}^{n} (x_p - x_i); \tag{2}$$

for example,

$$f(x_0 x_1, x_2) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}. \tag{3}$$

An upper limit can be obtained for the value of a divided difference by means of the following theorem, see [1]. If $f(x)$ and its first $n-1$ derivatives are finite and continuous and the nth derivative $f^{(n)}(x)$ exists, then

$$f(x_0, x_1, \cdots, x_n) = \frac{f^{(n)}(\xi)}{n!}, \tag{4}$$

where ξ lies in the range between the smallest and largest of the arguments x_0 to x_n.

For a divided difference involving a repeated argument one takes the limit as two initially distinct arguments approach one another; thus

$$f(x_0, x_0) = \lim_{x_0' \to x_0} f(x_0', x_0) = \lim_{x_0' \to x_0} \left[\frac{f(x_0')}{x_0' - x_0} + \frac{f(x_0)}{x_0 - x_0'} \right] = \frac{d}{dx_0} f(x_0). \tag{5}$$

In general, as shown in [2],

$$f(\overbrace{x_0, \cdots, x_0}^{k_0+1}, \overbrace{x_1, \cdots, x_1}^{k_1+1}, \overbrace{x_n, \cdots, x_n}^{k_n+1}) = \frac{1}{k_0! k_1! \cdots k_n!} \frac{\partial k_0 + k_1 + \cdots + k_n}{\partial x_0^{k_0} \partial x_1^{k_1} \cdots \partial x_n^{k_n}} f(x_0, x_1, \cdots, x_n) \tag{6}$$

and, for the special case $k_i = k$, for all i,

$$(7)\quad F_k \equiv f(\overbrace{x_0, \cdots, x_0}^{k+1}, \overbrace{x_1, \cdots, x_1}^{k+1}, \cdots, \overbrace{x_n, \cdots, x_n}^{k+1}) = \frac{1}{(k!)^{n+1}}\left[\prod_{i=0}^{n}\frac{\partial^k}{\partial x_i^k}\right]\sum_{p=0}^{n}\frac{f(x_p)}{\Pi_p},$$

where Π_p is given by (2).

From (7)

$$(8)\quad F_k = \frac{1}{(k!)^{n+1}}\sum_{p=0}^{n}\left[\prod_{i=0}^{n}\frac{\partial^k}{\partial x_i^k}\right]\frac{f(x_p)}{\Pi_p} = \frac{1}{(k!)^{n+1}}\sum_{p=0}^{n}\frac{\partial^k}{\partial x_p^k}\left\{f(x_p)\left[\prod_{\substack{i=0\\ i\neq p}}^{n}\frac{\partial^k}{\partial x_i^k}\right]\frac{1}{\Pi_p}\right\},$$

and since in (8)

$$(9)\quad \frac{\partial^k}{\partial x_i^k}\frac{1}{\Pi_p} = \frac{k!}{(x_p - x_i)^k}\frac{1}{\Pi_p}, \quad \text{for} \quad i \neq p,$$

one has

$$(10)\quad F_k \equiv f(\overbrace{x_0, \cdots, x_0}^{k+1}, \cdots, \overbrace{x_n, \cdots, x_n}^{k+1}) = \frac{1}{k!}\sum_{p=0}^{n}\frac{\partial^k}{\partial x_p^k}\frac{f(x_p)}{\Pi_p^{k+1}}.$$

Therefore by Leibnitz's Theorem

$$(11)\quad f(\overbrace{x_0, \cdots, x_0}^{k+1}, \cdots, \overbrace{x_n, \cdots, x_n}^{k+1}) = \sum_{p=0}^{n}\sum_{s=0}^{k} N_p{}^s f^{(s)}(x_p),$$

where $f^{(s)}(x_p)$ represents the sth derivative of $f(x_p)$ and

$$(12)\quad N_p{}^s \equiv \frac{1}{s!(k-s)!}\frac{\partial^{k-s}}{\partial x_p{}^{k-s}}\frac{1}{\Pi_p{}^{k+1}}.$$

Special Case $k = 1$

From (12)

$$(13)\quad N_p{}^0 = \frac{\partial}{\partial x_p}\Pi_p{}^{-2} = -2\Pi_p'\Pi_p{}^{-3}$$

$$N_p{}^1 = \Pi_p{}^{-2}$$

where

$$(14)\quad \Pi_p' \equiv \frac{\partial \Pi_p}{\partial x_p} = \Pi_p \sum_{\substack{j=0\\ j\neq p}}^{n}\frac{1}{x_p - x_j}.$$

For equally spaced arguments $x_i = x_0 + ih$, where $i = 0, 1, \cdots, n$, one has from (2)

$$(15)\quad \Pi_p = (-1)^{n-p}p!(n-p)!h^n.$$

Likewise, from (14)

$$\Pi_p' = (-1)^{n-p} p!(n-p)![S_p - S_{n-p}]h^{n-1}, \tag{16}$$

where $S_0 = 0$ and

$$S_r = \sum_{i=1}^{r} \frac{1}{i}, \quad \text{for} \quad r \neq 0; \tag{17}$$

therefore, from (13)

$$N_p^0 = \frac{-2[S_p - S_{n-p}]}{h^{2n+1}[p!(n-p)!]^2}$$
$$N_p^1 = \frac{1}{h^{2n}[p!(n-p)!]^2}. \tag{18}$$

Multiplying (11) through by $(n!)^2 h^{2n+1}$ and making use of (4), one has

$$\sum_{p=0}^{n} A_{np} f(x_p) = h \sum_{p=0}^{n} B_{np} f^{(1)}(x_p) - \frac{h^{2n+1}}{D_n} f^{(2n+1)}(\xi), \tag{19}$$

where $x_0 < \xi < x_n$, and one sets

$$A_{np} = 2[S_p - S_{n-p}][C_p^n]^2 \tag{20}$$

$$B_{np} = [C_p^n]^2 \tag{21}$$

and

$$D_n = \frac{(2n+1)!}{(n!)^2}. \tag{22}$$

In the above equations

$$C_p^n = \frac{n!}{p!(n-p)!} \tag{23}$$

are the binomial coefficients.

The first two quadrature formulas corresponding to $n = 1$ and $n = 2$ are the trapezoidal rule and Simpson's rule, respectively, applied to the derivative of $f(x)$. The formulas for higher value of n have the usually undesirable feature of involving the value of the integrated function $f(x)$ at intermediate values of x between x_0 and x_n. They should, however, due to their very small remainder terms, prove useful in checking the accuracy of integration in the numerical solution of differential equations. Moreover, these quadrature formulas should be advantageous in the solution of two-point boundary value problems, since these must be solved by a relaxation or by an iteration method.

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1. L. M. MILNE-THOMSON, *The Calculus of Finite Differences*, Macmillan & Company, London, 1933, p. 5–6.
2. *Ibid.*, p. 14.

Tables of the coefficients in (19) are given below:

TABLE 1

Coefficients A_{np}

n \ p	0	1	2	3	4	5	6
1	-2	2					
2	-3	0	3				
3	$-\frac{11}{3}$	-9	9	$\frac{11}{3}$			
4	$-\frac{25}{6}$	$-\frac{80}{3}$	0	$\frac{80}{3}$	$\frac{25}{6}$		
5	$-\frac{137}{30}$	$-\frac{325}{6}$	$-\frac{200}{3}$	$\frac{200}{3}$	$\frac{325}{6}$	$\frac{137}{30}$	
6	$-\frac{49}{10}$	$-\frac{462}{5}$	$-\frac{525}{2}$	0	$\frac{525}{2}$	$\frac{462}{5}$	$\frac{49}{10}$

TABLE 2

Coefficients B_{np}

n \ p	0	1	2	3	4	7	6	D_n
1	1	1						6
2	1	4	1					30
3	1	9	9	1				140
4	1	16	36	16	1			630
5	1	25	100	100	25	1		2772
6	1	36	225	400	225	36	1	12012

On a Device for Computing the $e_m(S_n)$ Transformation

1. Introduction. The transformation

$$(1)\qquad e_m(S_n) = \frac{\begin{vmatrix} S_n & S_{n+1} & \cdots & S_{n+m} \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+m-1} & & \cdots & \Delta S_{n+2m-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+m-1} & & \cdots & \Delta S_{n+2m-1} \end{vmatrix}}$$

has been developed by Schmidt [1] and Shanks [2]. It is shown by both authors that

$$e_m(S_n) = a$$

if

$$(2)\qquad S_n = a + \sum_{r=1}^{m} b_r \epsilon_r^n.$$

The former writer shows that iterates of the form (2) occur in the Gauss-Seidel method for the solution of linear equations, and derives the condition

$$(3)\qquad \begin{vmatrix} \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+m+1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+m} & \Delta S_{n+m+1} & \cdots & \Delta S_{n+2m} \end{vmatrix} = 0$$

for the applicability of the transformation.

A consideration which militates against the use of the transformation is that the vast amount of labor spent in evaluating the determinants in (1) serves only to produce one transformed result. The computation of further results proceeds quite independently with a similar expenditure of effort, so that use of (1) in its present form as a sequence to sequence transformation in which members of the transformed sequences $e_m(S_n)(n = \bar{n}, \bar{n}+1, \cdots, m = 1, 2, \cdots)$ may be examined for any tendency to approach a limit, involves a prohibitive expenditure of labor. It is proposed to show that the transformation (1) may be effected by a simple algorithm, in which transformed results for $n = \bar{n}, \bar{n}+1, \cdots, m = 1, 2, \cdots$ are progressively available for comparison.

2. Theorem. *If*

$$\epsilon_{2m}(S_n) = e_m(S_n)$$

and

$$\epsilon_{2m+1}(S_n) = \frac{1}{e_m(\Delta S_n)}$$

then

$$(4)\qquad \epsilon_{s+1}(S_n) = \epsilon_{s-1}(S_{n+1}) + \frac{1}{\epsilon_s(S_{n+1}) - \epsilon_s(S_n)}, \quad s = 1, 2, \cdots,$$

provided that none of the quantities $\epsilon_{2m}(S_n)$ becomes infinite.

Proof. The above result is easily verified when $s = 1$, for:

$$\epsilon_0(S_{n+1}) + \frac{1}{\epsilon_1(S_{n+1}) - \epsilon_1(S_n)} = S_{n+1} + \frac{1}{\dfrac{1}{\Delta S_{n+1}} - \dfrac{1}{\Delta S_n}}$$

$$= \frac{S_{n+1}(\Delta S_n - \Delta S_{n+1}) + \Delta S_{n+1} \cdot \Delta S_n}{\Delta S_n - \Delta S_{n+1}}$$

$$= \frac{S_n \Delta S_{n+1} - S_{n+1} \cdot \Delta S_n}{\Delta S_{n+1} - \Delta S_n}$$

$$= \epsilon_2(S_n).$$

For larger values of s, consider first the case $s = 2k$. It is then to be proved that

$$(5) \quad \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta^2 S_n & \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta^2 S_{n+k-1} & & \cdots & \Delta^2 S_{n+2k-1} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta^2 S_{n+k-1} & & \cdots & \Delta^2 S_{n+2k-2} \end{vmatrix}}{\begin{vmatrix} \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix} - \begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} \\ \Delta S_{n+2} & \Delta S_{n+3} & \cdots & \Delta S_{n+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}}$$

$$= \frac{1}{\begin{vmatrix} S_{n+1} & S_{n+2} & \cdots & S_{n+k+1} \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix} \begin{vmatrix} S_n & S_{n+1} & \cdots & S_{n+k} \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}} - \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k-1} & & \cdots & \Delta S_{n+2k-1} \end{vmatrix}}.$$

Rearranging columns and rows, the left hand side of equation (5) may be written

$$\frac{\begin{vmatrix} 1 & 1 & \cdots & & 1 \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} & \Delta^2 S_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k-1} & \Delta^2 S_{n+k-1} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta^2 S_{n+k-1} & & \cdots & \Delta^2 S_{n+2k-2} \end{vmatrix}}{\begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} & \Delta S_n \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} & \Delta^2 S_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k-1} & \Delta^2 S_{n+k-1} \end{vmatrix} - \begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta^2 S_{n+k-1} & \Delta^2 S_{n+k} & \cdots & \Delta^2 S_{n+2k-2} \end{vmatrix}},$$

and by the Schweinsian expansion of the quotient of two determinants [3] this expression is equivalent to

$$(6)\quad \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} & \Delta S_n \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} & \Delta^2 S_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta^2 S_{n+k-1} & \Delta^2 S_{n+k} & \cdots & \Delta^2 S_{n+2k-2} & \Delta^2 S_{n+k-2} \end{vmatrix} \begin{vmatrix} \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} \\ \Delta^2 S_{n+2} & \Delta^2 S_{n+3} & \cdots & \Delta^2 S_{n+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k-1} \end{vmatrix}}{\begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} & \Delta S_n \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} & \Delta^2 S_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k-1} & \Delta^2 S_{n+k-1} \end{vmatrix} \begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta^2 S_{n+k-1} & \Delta^2 S_{n+k} & \cdots & \Delta^2 S_{n+2k-2} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix}}{\begin{vmatrix} \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix} \begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} \\ \Delta S_{n+2} & \Delta S_{n+3} & \cdots & \Delta S_{n+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}}.$$

The right hand side of equation (5) may be written

$$(7)\quad (-1)^k \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix}}{\begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \\ S_{n+1} & S_{n+2} & \cdots & S_{n+k+1} \end{vmatrix} \begin{vmatrix} \Delta S_{n+1} & \cdots & \Delta S_{n+k} & \Delta S_n \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & \Delta S_{n+k-1} \\ 1 & \cdots & 1 & 1 \end{vmatrix}}$$

$$- \frac{\begin{vmatrix} \Delta S_{n+1} & \cdots & \Delta S_{n+k} & \Delta S_n \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & \Delta S_{n+k-1} \\ S_{n+1} & \cdots & S_{n+k} & S_n \end{vmatrix} \begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \\ 1 & 1 & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} \Delta S_{n+1} & \cdots & \Delta S_{n+k} & \Delta S_n \\ \cdots & \cdots & \cdots & \cdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & \Delta S_{n+k-1} \\ 1 & \cdots & 1 & 1 \end{vmatrix}}$$

and by using an extensional identity derived from

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} |a_1| & |b_2| - |b_1| & |a_2| \end{vmatrix}$$

(7) is equivalent to

$$(-1)^k \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} \end{vmatrix}$$

$$\begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} & \Delta S_{n+k+1} & \Delta S_n \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k-1} & \Delta S_{n+2k} & \Delta S_{n+k-1} \\ S_{n+1} & S_{n+2} & \cdots & S_{n+k} & S_{n+k+1} & S_n \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k} \\ \Delta S_{n+2} & \Delta S_{n+3} & \cdots & \Delta S_{n+k+1} \\ \cdot & \cdot & \cdots & \cdot \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}$$

which is equivalent to expression (6). Thus the left and right hand sides of (4) are equal when $s = 2k$. That they are also equal when $s = 2k+1$ is demonstrated in a similar fashion.

3. Algorithm. It can now be seen that the quantities S_n, $\epsilon_s(S_n)$; $n = \bar{n}, \bar{n}+1, \cdots$; $s = 1, 2, \cdots$; may be arranged as in a table of S_n and its differences, in which $\epsilon_s(S_n)$ takes the place of $\Delta^s S_n$. This table is composed of groups of four quantities $\epsilon_s(S_n)$, $\epsilon_s(S_{n+1})$, $\epsilon_{s-1}(S_{n+1})$, $\epsilon_{s+1}(S_n)$, arranged in the form

$$\begin{matrix} & \epsilon_s(S_n) & \\ \epsilon_{s-1}(S_{n+1}) & & \epsilon_{s+1}(S_n). \\ & \epsilon_s(S_{n+1}) & \end{matrix}$$

The quantity $\epsilon_{s+1}(S_n)$ is formed by the relation

$$\epsilon_{s+1}(S_n) = \epsilon_{s-1}(S_{n+1}) + \frac{1}{\epsilon_s(S_{n+1}) - \epsilon_s(S_n)},$$

and it may be remarked that this simple calculation is easily programmed for a digital computer, all the entries in the second and subsequent columns being obtained from this programme by trivial alterations in storage parameters.

In this table the even order columns $\epsilon_{2r}(S_n)$ display the transformed results $e_r(S_n)$. There is no need for the explicit evaluation of the determinant in equation (3), as it occurs in the table as the denominator of $\epsilon_{2m+1}(S_n)$; if equation (3) holds when $n = \bar{n}, \bar{n}+1, \cdots$, then $\epsilon_{2m}(S_{\bar{n}}) = \epsilon_{2m}(S_{\bar{n}+1}) = \cdots$.

4. A numerical illustration. A numerical illustration is provided by the transformation of the sequence produced by the linearly convergent iterative scheme

$$S_{n+1} = \tfrac{1}{4}(S_n^2 + 2)$$

which may be used to derive the smaller zero of the Laguerre polynomial $L_2(x) = x^2 - 4x + 2$, i.e., $x = 0.58578\ 64375$.

It will be noted from Table I (and this is a consistent feature of the transformation when meeting with numerical success), that

$$\left|\frac{1}{\epsilon_{2s}(S_{m+1}) - \epsilon_{2s}(S_m)}\right| \gg |\epsilon_{2s-1}(S_{m+1})|$$

and hence

$$\epsilon_{2s+2}(S_m) \doteq \frac{1}{\dfrac{1}{\epsilon_{2s}(S_{m+2}) - \epsilon_{2s}(S_{m+1})} - \dfrac{1}{\epsilon_{2s}(S_{m+1}) - \epsilon_{2s}(S_m)}} + \epsilon_{2s}(S_{m+1})$$

$$= \epsilon_{2s}(S_{m+1}) - \frac{[\epsilon_{2s}(S_{m+2}) - \epsilon_{2s}(S_{m+1})][\epsilon_{2s}(S_{m+1}) - \epsilon_{2s}(S_m)]}{[\epsilon_{2s}(S_{m+2}) - 2\epsilon_{2s}(S_{m+1}) + \epsilon_{2s}(S_m)]}.$$

It is thus an approximate rule that there are as many significant figures in the difference $\epsilon_{2s+2}(S_m) - \epsilon_{2s}(S_{m+1})$ as there are in the second difference $\epsilon_{2s}(S_{m+2}) - 2\epsilon_{2s}(S_{m+1}) + \epsilon_{2s}(S_m)$. From this it is seen that loss of figures due to cancellation may occur in the transformation. This possible loss of figures in the Aitken transformation $e_1(S_n)$ has been remarked upon by Olver [4] and Shanks [5].

Table I also displays terms in the transformed sequence $e_1^2(S_n)$, and it will

TABLE I

n	S_n	ϵ_1	ϵ_2	ϵ_3	ϵ_4	e_1^2
0	0.0					
		2.0				
1	0.5		0.57142 85714			
		16.0		89.11111 08		
2	0.5625		0.58510 63830		0.58574 34871	0.58576 19630
		60.2352 941		1658.713		
3	0.57910 15625		0.58573 19781		0.58578 57313	0.58578 61706
		211.05540		20262.3		
4	0.58383 96549		0.58578 18504		0.58578 64208	0.58578 64316
		725.9366		239060.		
5	0.58521 71856		0.58578 60462			
		2483.836				
6	0.58561 97886					

be noted that comparison of the results given by the transformations $e_2(S_n)$ and $e_1^2(S_n)$ in this case reacts slightly in favor of the latter. The difference between the two results however is not greatly significant, and there seems to be some point in using the transformation $e_m(S_n)$ consistently (in place of m successive applications of Aitken's transformation) just in case S_n might very accurately be represented by some equation of the form (2). There is no reason to believe that this is true for the example chosen, but results for a favorable numerical example are given in Schmidt's paper. Further examples in which success for $e_m(S_n)$ might be anticipated from analytical knowledge of S_n are given by Shanks.

5. An application. Having shown that the various transformations $e_m(S_n)$ may be effected without much effort, the primary purpose of this paper has been achieved. Functions of the form (2) are of such widespread occurrence in the theory of Mathematical Physics, that an exhaustive account of the problems in

which the transformation $e_m(S_n)$ might prove useful (besides being irrelevant to the main thesis of this paper which is primarily one of computational expediency) would be excessively discursive and, in view of possible developments, necessarily incomplete. By way of a footnote however it may be pointed out that the transformation may be applied to establish a criterion for the fitting of certain types of statistical data. The determination of the constants a_i, b_i when fitting tabular data, which is given at regularly spaced intervals in t of magnitude w, to a function of the form

$$\phi(t) = \sum_{i=1}^{h} b_i e^{-a_i t}, \tag{8}$$

when m is given, is already well known [6]. To establish the suitability of (8) as an adequate representation of the data, and further to determine h, it is merely necessary to apply the transformation $e_m(S_n)$ to the data, writing $S_{n+r} = \phi(t + rw)$, when $e_h(S_n) = 0$. (Various refinements, such as partitioning the data in groups of p values, and so on, are obvious and need not be discussed here.) This was one of the first problems upon which the transformation $e_m(S_n)$ [5] was used.

It is the author's hope that by demonstrating the ease with which the various transformations may be effected, their field of application might be widened, and deeper insight thereby obtained into the problems for whose solution the transformations have been used.

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1. J. R. SCHMIDT, "On the numerical solution of linear simultâneous equations by an iterative method," *Phil. Mag.*, Ser. 7, v. 32, 1941, p. 369–383.
2. D. SHANKS, "Non-linear transformations of divergent and slowly convergent sequences," *J. Math. and Physics*, v. 34, 1955, p. 1–42.
3. A. C. AITKEN, *Determinants and Matrices*, Oliver and Boyd, 6th Ed., 1949, Ch. V, p. 108, formula (2) and p. 49, formula (9).
4. F. W. J. OLVER, "The evaluation of zeros of high degree polynomials," Roy. Soc. Phil., *Trans.*, v. 244, 1952, p. 385–415.
5. D. SHANKS, "An analogy between transients and mathematical sequences and some non-linear sequence-to-sequence transforms suggested by it," Part I. Naval Ordnance Laboratory Memorandum 9994, White Oak, Md., 1949.
6. E. T. WHITTAKER & G. ROBINSON, *The Calculus of Observations*, Blackie and Son Ltd., 4th Ed., Ch. XV, 1944, p. 369.

TECHNICAL NOTES AND SHORT PAPERS

Note on the Computation of Certain Highly Oscillatory Integrals

The purpose of this note is to draw attention to the possible use of the Faltung theorem for Fourier transforms as an aid to the computation of highly oscillatory integrals.

If F is the Fourier cosine transform of f, defined by

$$F(u) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos ut \, dt$$

and G is the Fourier cosine transform of g, then

$$\int_0^\infty f(t)g(t)\cos ut\,dt = \frac{1}{2}\int_0^\infty F(y)[G(u+y)+G(|u-y|)]dy.$$

Numerical quadrature of the integral on the left will be laborious if u is large. But, if the transforms F, G are known, the integral on the right may prove much more tractable. The essential reason for this is that the large parameter u now appears in the argument of G as an additive constant and not as a multiplicative constant as in the integral on the left.

As an example, consider the integral

$$I = \int_1^\infty (t^2-1)^{-\frac{1}{2}} e^{-at^2}\cos bt\,dt$$

for large values of b. Here

$$f(t) = e^{-at^2} \qquad F(u) = \frac{1}{\sqrt{(2a)}}e^{-u^2/4a}$$

$$g(t) = (t^2-1)^{-\frac{1}{2}} \quad \text{for} \quad t>1 \qquad G(u) = -\sqrt{\frac{\pi}{2}}\,Y_0(u)$$

$$= 0 \quad \text{for} \quad t \le 1$$

so that

$$I = -\frac{1}{4}\sqrt{\frac{\pi}{a}}\int_0^\infty e^{-y^2/4a}[Y_0(b+y)+Y_0(|b-y|)]dy.$$

This form is, for a wide range of values of a and b, very suitable for computation using the simple trapezoidal rule [1,2]. The method was used very successfully in preparing a table of an integral, closely related to the one above, which occurs in the theory of the wave resistance of ships [UMT 174, *MTAC*, v. 7, 1953, p. 248].

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1. E. T. GOODWIN, "The evaluation of integrals of the form $\int_{-\infty}^{\infty} f(x)e^{-x^2}dx$," Camb. Phil. Soc., *Proc.*, v. 45, 1949, p. 241–245.
2. HENRY E. FETTIS, "Numerical calculation of certain definite integrals by Poisson's Summation Formula," *MTAC*, v. 9, 1955, p. 85–91.

On a Cubically Convergent Process for Determining the Zeros of Certain Functions

An iterative procedure of the form $x_{n+1} = f(x_n)$ for the determination of the roots of the equation $\phi(x) = 0$ may be classified as linearly convergent, quadratically convergent, and so on, by using a notation due to Hartree [1]: If $x = X$

is a solution of $\phi(x) = 0$, $x_n = X + \eta_n$ (so that η_n is the error in the determination of the root at each stage) and η_n is sufficiently small, then the convergence of the procedure $x_{n+1} = f(x_n)$ is said to be of the k-th order if $\eta_{n+1} = 0(\eta_n{}^k)$. This means in effect that when using a quadratically convergent procedure we double the number of known figures in the solution at each stage, when using a cubically convergent procedure we treble this number of figures, and so on. An example of a quadratically convergent procedure is provided by the Newton-Raphson process $x_{n+1} = x_n - \dfrac{\phi(x_n)}{\phi'(x_n)}$ (where primes denote differentiation with respect to x). It may be shown that for this process $\lim_{n\to\infty} \dfrac{\eta_{n+1}}{\eta_n{}^2} = \dfrac{1}{2}\dfrac{\phi''(X)}{\phi'(X)}$, so that the Newton-Raphson process is indeed quadratically convergent.

Various cubically convergent processes for extracting the roots of an equation are known, but they are not generally used because they involve the determination of the second derivative $\phi''(x_n)$ and the labor in doing so is normally considered to offset the advantage of gaining the extra figures. It is the purpose of this note to point out that if the function $\phi(x)$ satisfies a differential equation of the form

$$p\phi'' + q\phi' + r\phi = s \tag{1}$$

(where p, q, r, and s are functions of x), the determination of $\phi''(x_n)$ is a trivial matter, once $\phi(x_n)$ and $\phi'(x_n)$ have been calculated. The number of known figures in X may thus be trebled at each stage with scarcely more effort than that necessary to effect the Newton-Raphson process. This remark applies equally well of course if $\phi(x)$ satisfies a nonlinear second order differential equation, but the form of the latter permits of greater variety and hence the subsequent analysis is more complex.

Perhaps the simplest known cubically convergent process is that due to H. W. Richmond [2], viz.,

$$x_{n+1} = x_n - \frac{2\phi(x_n)\phi'(x_n)}{2(\phi'(x_n))^2 - \phi(x_n)\phi''(x_n)}. \tag{2}$$

It may be shown that for this process $\lim_{n\to\infty} \dfrac{\eta_{n+1}}{\eta_n{}^3} = \dfrac{3(\phi''(X))^2 - 2\phi'(X)\phi'''(X)}{12(\phi'(X))^2}$

Substituting for $\phi''(x_n)$ in (2) from equation (1), we have

$$x_{n+1} = x_n - \frac{2\phi\phi' p}{2(\phi')^2 p - \phi s + r\phi^2 + q\phi\phi'} \tag{3}$$

where the argument $x = x_n$ on the right hand side of the equation (3) is understood. An illustration of the power of the method is afforded by the computation of the first three zeros of $J_0(x)$, which satisfies the differential equation

$$x\phi'' + \phi' + x\phi = 0,$$

i.e., $p = x$, $q = 1$, $r = x$, $s = 0$. For this problem ϕ' is easily determined from tables since $J_0'(x) = -J_1(x)$. Inspection of the tables [3] shows that zeros of $J_0(x)$ occur near $x = 2.405$, $x = 5.520$, and $x = 8.654$. Table I gives the results obtainable by one application of (3), and contrasts these with those obtained by one application of the Newton-Raphson process.

TABLE I

Initial approximation	Result of Newton-Raphson process	Result of Richmond's process
2.405	2.404 825 551	2.404 825 557 697
5.520	5.520 078 110	5.520 078 110 286
8.654	8.653 727 909	8.653 727 912 914

These results were derived solely from tabulated values of $J_0(x)$ and $J_1(x)$, and required no subsequent interpolation. The quantities in the last column are in error in the last figure by one, one and three units, respectively.

The process is easily extended to the extraction of complex roots. Changing to a new independent variable z, where $z = x + iy$, and letting primes now denote differentiation with respect to z, we have, after some manipulation

$$(4) \qquad y_{n+1} - y_n = \frac{DA - BC}{A^2 + B^2} \quad x_{n+1} - x_n = \frac{CA + DB}{A^2 + B^2},$$

where

$$\begin{aligned}
A &= 2p_{re}(\phi''_{re} - \phi''_{im}) - 4p_{im}\phi'_{re}\phi'_{im} - \phi_{re}s_{re} + \phi_{im}s_{im} + r_{re}(\phi^2_{re} - \phi^2_{im}) \\
&\quad - 2r_{im}\phi_{re}\phi_{im} + q_{re}(\phi_{re}\phi'_{re} - \phi_{im}\phi'_{im}) - q_{im}(\phi_{re}\phi'_{im} + \phi_{im}\phi'_{re}) \\
B &= 4p_{re}\phi'_{re}\phi'_{im} + 2p_{im}(\phi''_{re} - \phi''_{im}) - \phi_{im}s_{re} - \phi_{re}s_{im} + 2r_{re}\phi_{re}\phi_{im} \\
&\quad + r_{im}(\phi^2_{re} - \phi^2_{im}) + q_{im}(\phi_{re}\phi'_{re} - \phi_{im}\phi'_{im}) + q_{re}(\phi_{re}\phi'_{im} + \phi_{im}\phi'_{re}) \\
C &= 2p_{re}(\phi_{re}\phi'_{re} - \phi_{im}\phi'_{im}) + 2p_{im}(\phi_{re}\phi'_{im} + \phi_{im}\phi'_{re}) \\
D &= -2p_{im}(\phi_{re}\phi'_{re} - \phi_{im}\phi'_{im}) - 2p_{re}(\phi_{re}\phi'_{im} + \phi_{im}\phi'_{re})
\end{aligned}$$

and the real and imaginary parts of $p(z_n)$ are written p_{re} and p_{im}, and so on.

An example of the application of the method in the complex case is given by the determination of a zero of $Y_0(z)$, which satisfies the differential equation

$$z\phi'' + \phi' + z\phi = 0;$$

i.e., we have $p_{re} = x_n$, $p_{im} = y_n$; $q_{re} = 1$, $q_{im} = 0$; $r_{re} = x_n$, $r_{im} = y_n$; $s_{re} = s_{im} = 0$; and $Y_0'(z) = -Y_1(z)$. Estimation from the tables [4,5] shows that a zero of $Y_0(z)$ occurs near $z = -2.3762 + i0.6367$. (The large number of figures in the initial approximation is due to the fact that in these tables $Y_0(z)$ is tabulated against the variables r and θ, where $z = re^{i\theta}$.) Table II displays the results given by the process (4) and contrasts this with the result of one application of the Newton-Raphson process modified for the determination of complex roots, viz.,

$$x_{n+1} - x_n = -\frac{\phi_{re}\phi'_{re} + \phi_{im}\phi'_{im}}{\phi''^2_{re} + \phi''^2_{im}} \quad y_{n+1} - y_n = \frac{\phi_{re}\phi'_{im} - \phi_{im}\phi'_{re}}{\phi''^2_{re} + \phi''^2_{im}}$$

TABLE II

Initial Approximation		Newton-Raphson process	Richmond's process	Correct result
x	−2.3762	−2.4047	−2.4031	−2.4030
y	0.6367	0.5407	0.5398	0.5399

It will be noted that this root is not as accurately determined as were the real roots of $J_0(x) = 0$. This is not essentially due to the fact that the computations take place in the complex plane, but due to a secondary consequence of this, namely that the initial approximation to the root is more inexact.

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1. D. R. HARTREE, "Notes on iterative processes," Cambridge Phil. Soc., *Proc.*, v. 45, 1949, p. 230–236; see also E. SCHRÖDER, "Über unendlich viele Algorithmen zür Auflösung der Gleichungen," *Math. Ann.*, v. 2, 1870, p. 317–363. The possible ambiguity in our definition of convergence of order k is not significant in the present context.
2. H. W. RICHMOND, "On certain formulae for numerical approximation," London Math. Soc., *Jn.*, v. 19, 1944, p. 31–38.
3. HARVARD UNIVERSITY, *Tables of the Bessel Functions of the First Kind of Orders Zero and One*, v. 3, Harvard Univ. Press, Cambridge, 1947.
4. COLUMBIA UNIVERSITY, *Table of the Bessel Functions $Y_0(z)$ and $Y_1(z)$ for Complex Arguments*, Columbia Univ. Press, New York, 1950.
5. COLUMBIA UNIVERSITY, *Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments*, Columbia Univ. Press, New York, 1943.

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

33[A].—EDGAR KARST, *A simple octal (binary) multiplication method for checking computer results.* 4 leaves, handwritten, deposited in the UMT FILE.

In this octal multiplication method, sums of one-digit products are accumulated mentally, with two intermediate lines being written.

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34[A-F, HI, K-N, P-V, Z].—KARL SCHÜTTE, *Index Mathematischer Tafelwerke und Tabellen (Index of Mathematical Tables)*, R. Oldenbourg, München, 1955, 143, p., 21 cm. Canvas cover. Price, DM 14.50.

This is a booklet of 143 pages which consists of the following parts: A list of approximately 1,200 tables and other useful books classified under 16 general headings, an index of authors of these books and tables, and an index of institutes referred to in the first part.

The headings of the 16 classes are as follows: I. Numerical and practical calculating, II. Logarithms of natural numbers, III. Logarithms of circular functions, IV. Natural values of circular functions, V. Simple functions derived from elementary functions, VI. Primes, prime factors, factors, compound interest and rent; continued fractions, theory of numbers, VII. Factorials, gamma functions, exponential and hyperbolic functions; elementary transcendental functions, VIII.

Elliptic functions and integrals, spherical, Bessel and other higher functions, IX. Integral tables and other higher functions; numerical solution of equations and differential equations, X. Tables applicable to physics, chemistry, and other sciences, XI. Astronomy and astrophysics, XII. Geodesy and geophysics, XIII. Nautical and aeronautical determination of position, XIV. Meteorology, XV. Astronautics, XVI. Miscellaneous, formulas and tables of measures, weights, monetary units.

There is a preface in both English and German and the Table of Contents is also bilingual, as are the headings and subheadings of the first part. In addition, there is a list of the abbreviations used in the book.

The avowed purpose of this volume is essentially the same as that of the earlier *Index of Mathematical Tables*, written by Fletcher, Miller, and Rosenhead, (*MTAC*, 1946–47, v. 2, p. 13–18, 136, 178–181, 219–220, and 277–8). The book contains material dated as late as 1955. However, the author states that the work was started before World War II and subsequently extended to 1955. Thus, it presumably covers 11 years (or almost 11 years) of publication time since the publication of Fletcher, Miller, and Rosenhead.

The 16 large headings are broken down into a total of 130 subdivisions. For example, category III. Logarithms of circular functions, has 2 major headings, namely, A. Sexagesimal arguments, and B. Centesimals (grades), Sexagesimal and centesimal arguments. Under A, the subheadings are degrees, time, degree and time, decimals of degree, and decimals of degree and time. Under heading B, the subheadings are Centesimal arguments (grades), and Sexagesimal and centesimal arguments (grades).

Within each of the 130 subdivisions (as appropriate), the titles are arranged in increasing order of the number of decimal places contained therein and within each group where the number of decimal places remains fixed the titles are arranged in chronological order.

Of the approximately 112 pages of indexed tables and books, 42 of them represent tables and books in categories 10 through 15, i.e., tables applicable to physics, chemistry, biochemistry, biology, paleozoology, technics and engineering, astronomy and astrophysics, geodesy and geophysics, nautical and aeronautical determination of position, meteorology and astronautics. Examination of these 7 categories leaves the impression that most of the tables indexed hereunder are "nonmathematical" in character, i.e., handbooks and special technical tables and the like. If the number of titles is approximately proportional to the number of pages, this means that only about 750 of the approximately 1200 indexed titles are "mathematical" in nature.

The approximately 450 titles in categories 10–15 are obviously useful information and probably such an index has not heretofore appeared in print. Certainly these categories are largely noncompetitive with the earlier index of Fletcher, Miller, and Rosenhead.

With respect to the approximately 750 titles of a "mathematical" nature, a sampling shows that probably between 50 and 60 per cent of them are dated 1944 and earlier and hence are within the purview of the Fletcher, Miller, and Rosenhead index. A finer examination of a few categories which can be compared with the Fletcher, Miller, Rosenhead index (namely, natural logarithms, natural

circular functions (degrees and decimals), hyperbolic functions of real arguments and their logarithms, elliptic functions and integrals, Bessel functions of real arguments) shows that in the 1944 and earlier category, Schütte has 42 titles as opposed to at least 140 titles in Fletcher, Miller, and Rosenhead. Furthermore, of the 42 titles appearing in Schütte in these categories, 31 also appear in Fletcher, Miller, and Rosenhead. If the percentages showing up in this sample persist, this would indicate that in the mathematical categories approximately three-fourths of the titles which are dated 1944 and earlier are duplicates of those appearing in Fletcher, Miller, and Rosenhead. The reviewer had no time to make a complete item count and does not know whether the particular categories picked for sampling are typical or not.

The reviewer is somewhat disappointed that many of the desirable features of Fletcher, Miller, and Rosenhead index are not included in this book. For example, there is no indication whether an item has actually been seen by the author. There is no indication of the interval or range of the arguments. There is also no indication of the facilities provided for interpolation in tables, or any explanation of the varying notation, etc. The "standard" or "classical" tables in a given category are not set forth in boldface type as they are in the F.M.R. index. It would seem also that some kind of decimal classification system would be desirable.

There are some relatively minor things which undoubtedly will be corrected in subsequent editions of the book. Some misspellings occur (p. 55, *change* DeLalla to DeLella; p. 83, *change* Salva to Salvo; p. 138, *change* J. M. Hammerley to J. M. Hammersley). In addition, there are probably a few incorrect titles (for example, on p. 55, the book listed as "8-figure values of the trigonometrical functions at interval 0°.01," by E. Buckingham probably is section 1 of the Manual of Gear Design written by Mr. Buckingham). A finer breakdown of categories would certainly be useful in many instances (for example, in the case of hyperbolic functions one might want to know which hyperbolic functions are tabulated in a particular place).

The book will undoubtedly serve a useful purpose since it does have material not available elsewhere. However, it is a little disappointing that the same care and precision did not go into its preparation as went into the F.M.R. Index, in favor of which the reviewer may very well be prejudiced.

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35[E].—NBS Applied Mathematics Series No. 45, *Table of Hyperbolic Sines and Cosines, $x = 2$ to $x = 10$*, U. S. Gov. Printing Office, Washington, D. C., 1955, v + 81 p., 25.7 cm. Price, 55 cents.

This table lists sinh x and cosh x for $x = 2(.001)10$, 9S.

It is an extension to AMS 36, *Tables of Circular Hyperbolic Sines and Cosines for Radian Arguments*, and its predecessor volume, Mathematical Tables 3, see Reviews 89, *MTAC*, v. 1, 1943, p. 45–7, and 33, *MTAC*, v. 9, 1955, p. 71–2, and note 114, *MTAC*, v. 4, 1950, p. 123–4. AMS 36 and its predecessors list, among other things, sinh x and cosh x for $x = 0(.0001)1.9999$, 9D.

There is adequate introduction. Here the addition formulas used in the computation of the table are stated, it is remarked that linear interpolation will yield at least six significant digits throughout, and quadratic interpolation formulas accurate to the full nine significant digits are stated.

In addition, it is remarked in the introduction that $\frac{1}{2}e^x$ is a 9S approximation for both sinh x and cosh x for $x > 10$.

In the introduction it is claimed that the error in tabulated values should be no more than a unit in the last place given. As a matter of routine, Mrs. R. B. Horgan and Mr. Edward Sallin recomputed the values for 200 randomly chosen arguments taken from [1]; they discovered 12 cases in which sinh x disagreed with the values in AMS 45 and 6 cases in which cosh x disagreed with the values computed in AMS 45. Their calculation was based on the values listed in the basic table by C. E. Van Orstrand [2]. The discrepancies are all 1 unit in the last place, which is within the accuracy claimed in AMS 45.

The table was computed on the National Bureau of Standards Eastern Automatic Computer (SEAC) by William F. Cahill. The results are transcribed onto punched cards, the values in these cards were then differenced, and the manuscript was prepared on the card-controlled typewriter at the National Bureau of Standards. The photographic reproduction is excellent, both in format and legibility.

The slight discrepancies noted by Mrs. Horgan and Mr. Sallin are tabulated below.

C. B. T.

TABLE 1

	sinh x	
x	sinh x from AMS 45	sinh x from SWAC
2.914	9.18805 728	9.18805 727
3.388	14.78645 18	14.78645 17
4.162	32.09210 88	32.09210 87
4.843	63.42072 31	63.42072 30
5.404	111.14465 9	111.14465 8
6.046	211.20880 0	211.20879 9
6.641	382.92958 3	382.92958 2
6.882	487.28626 7	487.28626 6
7.060	582.22215 4	582.22215 3
8.190	1802.36099	1802.36098
8.556	2598.92398	2598.92397
8.572	2640.84121	2640.84120

TABLE 2

	cosh x	
x	cosh x from AMS 45	cosh x from SWAC
4.617	50.59994 44	50.59994 43
5.101	82.09605 10	82.09605 09
5.914	185.09331 8	185.09331 7
8.510	2482.08165	2482.08164
8.824	3397.69603	3397.69602
9.247	5186.69945	5186.69944

1. The RAND Corporation, *A Million Random Digits with 100,000 Normal Deviates*, The Free Press, 1955, Review 11, *MTAC*, v. 10, 1956, p. 39–43.
2. C. E. Van Orstrand, "Tables of the exponential function and of the circular sine and cosine to radian argument," *Nat. Acad. Sci. Memoirs*, v. 14, 1921, Fifth memoir.

36[E].—NBS Applied Mathematics Series No. 46, *Table of the Descending Exponential*, $x = 2.5$ to $x = 10$. U. S. Gov. Printing Office, Washington, D. C., 1955, v + 76 p., 25.7 cm. Price, 50 cents.

This table lists e^{-x}, $x = 2.5(.001)10$, 20D. There is a short introduction, which describes the computation and printing of the table but otherwise refers the reader to an earlier volume [1].

Calculations were performed on the SEAC computer to 25D, rounded to 20D, differenced for checking (no differences are published), and typed by a card-controlled typewriter (presumably the IBM model 058 at the National Bureau of Standards) for photographic reproduction.

This publication is considered to be auxiliary to [1], which lists (among other things) e^{-x}, $x = 0(.0001)2.4999$, 18D. In the brief introduction the reader is referred to this earlier volume for bibliographic notes and other information.

As a matter of routine, Mrs. R. B. Horgan and Mr. Edward Sallin used the SWAC computer to check the values at 200 randomly chosen arguments (taken from [2]) against products of the values listed by Van Orstrand [3]; there were no discrepancies between the listed values and the values computed on SWAC.

The printing is easily legible and entirely satisfactory.

C. B. T.

1. National Bureau of Standards Applied Mathematics Series 14, *Tables of the Exponential Function e^x*, U. S. Government Printing Office, 1951, reviewed in an earlier edition, RMT 524, *MTAC*, v. 3, 1948–9, p. 173.
2. The RAND Corporation, *A Million Random Digits with 100,000 Normal Deviates*, The Free Press, 1955, Review 11, *MTAC*, v. 10, 1956, p. 39–43.
3. C. E. Van Orstrand, "Tables of the exponential function and of the circular sine and cosine to radian argument," *Nat. Acad. Sci. Memoirs*, v. 14, 1921, Fifth memoir.

37[G].—J. D. Swift, *Tables of Steiner Triple Systems*, 4 p., 28 cm., +7 p., 56 cm., manuscript. Deposited in UMT File.

These tables list one sytem of each class of isomorphic Steiner triple systems of orders 3, 7, 9, 13, and 15. Some algebraic and classification data are included in the listings.

There is only one class for each of the orders 3, 7, and 9; there are two classes of order 13; there are 80 classes of order 15.

The 4 small pages contain introductory and description material and the tables through those of order 13. The 7 large pages contain the table of the 80 classes of order 15.

Part of the material was computed by the author and a colleague in connection with their paper on Steiner triple systems of order 15 [1]. Material was also taken from a listing of systems order 13 by F. N. Cole [2]. Cross references are made to papers by A. S. White, F. N. Cole, and Louise B. Cummings [3], and R. A. Fisher [4].

C. B. T.

1. Marshall Hall, Jr., & J. D. Swift, "Determination of Steiner triple systems of order 15," *MTAC*, v. 9, 1955, p. 146–152.

2. F. N. COLE, "The triad systems of 13 letters," Am. Math. Soc., *Trans.*, 1913, v. 14, p. 1–5.
3. A. S. WHITE, F. N. COLE, & LOUISE D. CUMMINGS, "Complete classification of triad systems on 15 elements," *Nat. Acad. Sci., Memoirs*, v. 14, 1925, 2nd Memoir, 89 p.
4. R. A. FISHER, "An examination of the different possible solutions of a problem in incomplete blocks," *Ann. of Eugenics*, v. 10, 1940, p. 52–75.

38[G, H, I, J, L, M].—ANDREW D. BOOTH, *Numerical Methods*, Academic Press, New York, Butterworth, London, 1955, viii + 195 p., $8\frac{1}{2}'' \times 5\frac{1}{2}''$. Price, $6.00.

Dr. Booth has a reputation [1] in X-ray crystallography, in the design and construction of what his publisher [2] calls "authentic" calculating machines, and in their use, e.g., in language translation. Unfortunately this book is not up to his usual standards. There are many signs of hasty writing and referencing, careless proofreading, and unfamiliarity with current typographical conventions [3]; indeed Dr. Booth appears to be unsure of the initials of his wife and collaborator (p. 190, 191). In addition there are substantial inaccuracies at various levels.

Among the errors which have caught our eye we mention the following:

On p. 38, there is the standard classroom mistake of assuming that the mean value ξ, in the error term of a Maclaurin series for $f(x)$, is independent of x, and taking it outside the sign of integration.

On p. 59 and elsewhere we find error terms in approximate quadratures given as $Ah^n f^n_{\max}$ instead of $\theta A h^n |f^n|_{\max}$ with $0 \leqslant |\theta| \leqslant 1$.

On p. 92 there is the suggestion that every orthogonal matrix is symmetric.

We quote, without comment and with the author's spelling, the first sentence of section 7.12 entitled Monte Carlo Methods, and the footnote to which reference is made: "The somewhat intriguing title of this section is now applied to methods of numerical analysis which make use of the theory of Games [12]. ((12) von Neumann, J., and Morganstern, O., *The Theory of Games*, Princeton (1947)).''

It is dangerous to recommend methods for the solution of particular problems in numerical analysis, such as the inversion of matrices and the determination of their eigenvalues, without a knowledge of personnel, equipment available, and of the details of the problem. Our experience does not always coincide with that of Dr. Booth.

Nothing less than a complete revision of the book can make it acceptable in its purpose to reveal to final honors B.Sc. students at the University of London the basic mathematical principles of modern computation.

The chapter headings are: The Nature and Purpose of Numerical Analysis; Tabulation and Differences; Interpolation; Numerical Differentiation and Integration; The Summation of Series; The Solution of Ordinary Differential Equations; Simultaneous Linear Equations; Partial Differential Equations; Nonlinear Algebraic Equations; Approximating Functions; Fourier Synthesis and Analysis; Integral Equations.

J. T.

1. A. D. BOOTH, *Fourier Technique in X-ray Organic Structure Analysis*, Cambridge, 1948.
A. D. BOOTH & K. H. V. BOOTH, *Automatic Digital Calculators*, Butterworths Sci. Pub., London, 1953.
W. N. LOCKE & A. D. BOOTH, editors, *Machine Translation of Language*, John Wiley and Sons, New York, 1955.
2. *Nature*, Butterworths Scientific Publications Advertisement, v. 175, 1955, p. ccccxli.

3. For instance, on one opening, p. 62–63, we find *three* ways of indicating the tail of an infinite series: $+ \cdots$, $+$ *etc.*, $\cdots$; on p. 53 we find $+$ *etc.*, $\cdots$, $\cdots$ *etc.*

Again we find $O(x^r)$ signifying a polynomial of degree r; elsewhere we find $O(10)$.

The reviewer would like to include a disclaimer on the part of SEAC to the discovery of the fifteenth Mersenne prime. That $2^{1279}-1$ is prime was established by SWAC (see *MTAC*, 6, 1952, p. 205), not by SEAC, as stated on p. 2.

39[G, H].—*Proceedings of the Conference on Linear Programming, May, 1954*, FERRANTI Ltd., London, England, 1955, 92 mimeographed p., 32.3 cm. A limited number of copies are available free from Ferranti Ltd., 21 Portland Place, London W. 1, England.

Ferranti Ltd., which has been constructing high-speed automatic computers, issued this report of a conference on linear programming which it sponsored on May 4, 1954. Of the eight papers presented, two deal with computation, the remainder with applications. E. M. L. Beale outlines his "method of leading variables," previously reported in more detail in [1], and presents worksheets of a numerical example. D. G. Prinz describes experiences on the Manchester Computer with the Simplex Method, using the version of the method in which the entire tableau is transformed. He remarks that some difficulties due to round off occurred when numbers conceptually zero were not actually zero in the machine. These were eliminated by modifying the code so that the conceptual zeros were never entered. The paper also offers a brief heuristic justification for the fact that, characteristically, the simplex method takes fewer iterations than the conceivable upper bound (unknown, by the way), but the argument is not very convincing.

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1. E. M. L. BEALE, "An alternative method for linear programming," Camb. Phil. Soc., *Proc.*, v. 50, 1954, p. 513–523.

40[H].—MARK LOTKIN, "A set of test matrices," 10 p., 27 cm. Typescript deposited in the UMT FILE.

Table 1 contains inverses of A_n, $N = 1(1)10$. This requires 4 pages. Table 2 lists the matrices H_n, $N = 1(1)10$. This requires 5 pages. Table 3 lists characteristic polynomials $F_n(\lambda)$, $N = 1(1)6$. This requires about a third of a page.

The matrix A_n is defined in the author's paper [1], p. 153; H_n is defined in [1], p. 158; the characteristic polynomials are for the matrices A_n.

C. B. T.

1. MARK LOTKIN, "A set of test matrices," *MTAC*, v. 9, 1955, p. 153–161.

41[L].—HAROLD D. LARSEN, *Rinehart Mathematical Tables, Formulas and Curves*, Enlarged Edition, 1953, reprinted 1956. Rinehart & Co., Inc., New York, viii + 280 + 2 p., 21 cm. Price $2.50.

Seven tables, 28–34, have been added to the earlier edition (*MTAC*, v. 3, 1949, p. 466) as follows:

T. 28: Values and Common Logarithms of the Gamma Function, $\Gamma(n)$, 5D, $n = 1.01(.01)2.50$.

T. 29: The Probability Integral, $\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt$, 5D, $t = 0(.001)2(.01)3.09$.

T. 30: Values of Bessel Functions $J_0(x)$ and $J_1(x)$, 4D, $x = 0(.1)14.9$; also smallest six positive zeros of $J_0(x)$ and $J_1(x)$.

T. 31: Values of the Complete Elliptic Integrals,

$$K = \int_0^{\pi/2} (1 - \sin^2\theta \sin^2\phi)^{-\frac{1}{2}} d\phi,$$

$$E = \int_0^{\pi/2} (1 - \sin^2\theta \sin^2\phi) d\phi, \text{ 4D,}$$

$$\theta = 0°(1°)70°(0.5°)80°(0.2°)89°(0.1°)90°.$$

T. 32: Values of the Elliptic Integral of the First Kind,

$$F(k, \phi) = \int_0^{\phi} (1 - \sin^2\theta \sin^2\phi)^{-\frac{1}{2}} d\phi, \text{ 4D,}$$

$$\theta = 5°(5°)90°, \quad \phi = 1°(1°)90°.$$

T. 33: Values of the Elliptic Integral of the Second Kind,

$$E = (k, \phi) = \int_0^{\phi} (1 - \sin^2\theta \sin^2\phi)^{\frac{1}{2}} d\phi, \text{ 4D,}$$

$$\theta = 5°(5°)90°, \quad \phi = 1°(1°)90°.$$

T. 34: Square Root Divisors (for Calculating Square Roots to 5 Significant Figures).

The typography and composition of the tables are good and pleasing to the eye of the user. It is certainly appropriate to have tables of such things as the Incomplete Elliptic Integrals of the First and Second Kinds available among a collection of elementary tables.

Several relatively minor things may be worthy of correction in the next edition. In **T. 29** and **T. 32–3**, it is not necessary to use the same letter for the upper limit and the variable of integration. In **T. 32–3**, the functions actually tabulated are $F(\theta, \phi)$ and $E(\theta, \phi)$ respectively instead of those advertised, viz., $F(k, \phi)$ and $E(k, \phi)$. For completeness and use in interpolation, it would be nice to have included $\theta = 0°$ and $\phi = 0°$ among the arguments; however, these entries are easily computed and further, their inclusion would have used more space and, in the case of $\theta = 0°$, would probably have necessitated an additional page in each table.

T. 34 appears to be of doubtful utility. With the use of **Table 4** and "visual interpolation," one can usually obtain $\sqrt{N}$ to 4S. Following this with one division, Newton's method $\left[x_{K+1} = \frac{1}{2}\left(x_K + \frac{N}{x_K}\right)\right]$ gives 7S or 8S. However, the reviewer is willing to be convinced by the results of a "challenge match."

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42[L].—GERTRUDE BLANCH & IDA RHODES, "Table of characteristic values of Mathieu's equation for large values of the parameter," Washington Academy of Sciences, *Jn.*, v. 45, 1955, p. 166–196.

The tabulation presented in this paper complements the NBS tables (*MTAC*, v. 6, 1952, p. 29–30), and completes the tabulation of characteristic values of Mathieu's equation for all orders $r \leq 15$. The notations are the same as in the NBS tables, except that the auxiliary parameter t has been used in place of $s = 1/t^2$, and the quantities

$$\mathrm{Be}_r(t) = \mathrm{be}_r(s) - (2r + 1)/t, \quad \mathrm{Bo}_r(t) = \mathrm{bo}_r(s) - (2r - 1)/t$$

have been tabulated in place of the characteristic values.

The chief aim was a numerical description of the characteristic values for $r = 0(1)15$ and $s \geq 100$, or $t \leq .1$. The intervals have been so chosen that interpolation by Everett's formula with second and fourth modified central differences will give full accuracy; and the requisite differences are given in the tables. All entries are given to 8D, and for $t \geq .01$ all entries should be correct, but the authors are careful in pointing out that their method of checking made it impossible to detect a random error of two units of the last decimal place. For $t < .01$ ($s > 10^4$), the accuracy is considerably less, and even the seventh decimal place cannot be guaranteed. However, in this range, 6D in $\mathrm{B}_r(t)$ mean at least 9S in $b_r(t)$.

The authors have produced a very useful set of tables. It is to be hoped that numerical work on Mathieu functions will continue with emphasis, in the future, on functions of higher orders.

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43[Q].—ADMIRALTY HYDROGRAPHIC SUPPLIES ESTABLISHMENT, *Tables of Computed Altitude and Azimuth*, v. I: *Latitudes 0°–14° North and South*. Hydrographic Publication H.D. 486/1, Hydrographic Dept., Admiralty, London, England, 1952, xvi + 351 p., 29.5 cm. Each volume is priced at 30s. 0d. and copies may be obtained from Admiralty Chart Agents, the Admiralty Hydrographic Supplies Establishment, Creechbarrow House, Taunton, and from any Admiralty Chart and Chronometer depot.

The U. S. Hydrographic Office Publication No. 214 has been reproduced in Tables of Computed Altitude and Azimuth by the British Hydrographic Department. The original H.O. 214 is in nine volumes, each covering 10° of latitude. The regrouping of the new edition results in six volumes, each covering 15°. The tabular matter is identical in both, except that certain errors have been corrected in the British publication. (These are listed, for Vol. 1, on page xvi of the Introduction.)

The tables are designed to solve directly the spherical triangle of astronomy. Latitude, declination, and hour angle are the arguments, yielding altitude and azimuth of the object under observation. Computed altitudes less than 5° are

omitted, and no refraction is included in the tabulated values. Altitudes are printed to the nearest tenth of a minute of arc, azimuths to the nearest tenth of a degree.

Latitudes and hour angles are given at intervals of 1°; declinations at various intervals, but not less than 30′. The choice of declination values was made appropriate to the positions of the Sun, Moon, planets, and the principal navigation stars. Thus, the interval is uniformly 30′ up to 30° declination. Above this value there are places where very large intervals occur, e.g., between 63° and 69° declination. This might cause inconvenience in other applications of the tables than that for which they were designed.

As an aid to interpolation for the actual declination and local hour angle two auxiliary quantities Δd and Δt are given. These represent, respectively, the change of altitude in minutes of arc for one minute change in declination, and the change in altitude for a one minute change in hour angle. A multiplication table is printed on the end papers to facilitate interpolation.

A complete description is given of the use of the tables, consistent with navigation practice in the British Navy.

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44[Q].—ADMIRALTY HYDROGRAPHIC SUPPLIES ESTABLISHMENT, *Sight Reduction Tables for Air Navigation*, v. 1: *Selected Stars for Epoch 1955.0*, 1952, xi + 324 p., 29.5 cm.; v. 2: *Latitudes 0°–39°, Declinations 0°–29°*; v. 3: *Latitudes 40°–89°, Declinations 0°–29°*, 1953, x + 338 p., 29.5 cm. A.P. 3270. H.M. Nautical Almanac Office, Herstmonceux Castle, Sussex, England. Each volume is priced at 30s. 0d. and copies may be obtained from Admiralty Chart Agents, the Admiralty Hydrographic Supplies Establishment, Creechbarrow House, Taunton, and from any Admiralty Chart and Chronometer depot.

These three volumes represent a joint effort of the U. S. Nautical Almanac Office, and H. M. Nautical Almanac Office. They are issued in the United States as Hydrographic Office Publication No. 249.

Volume 1. Values of altitude, to the nearest minute of arc, and true azimuth, to the nearest degree are given for six selected stars for every degree of latitude and local hour angle of the first point of Aries. (Above latitude 70° the local hour angle is given at intervals of 2°.) The stars were chosen to be those best placed for observation at any given time and place. The tables are arranged so as to provide for a change in the choice of stars every hour, if necessary (every two hours, above 70° latitude). As the star positions are affected by proper motion of the stars themselves it is planned to issue a new edition of this volume every five years. No refraction has been applied to the tabulated altitudes.

These tables are of limited value outside their intended area of operation which is the planning of observations for aircraft flights, and their subsequent reduction. (Various auxiliary tables are given on the end papers. These include a table to aid interpolation, one for conversion of angles of arc to time, a refraction table, and a table of Coriolis corrections.)

The British edition is reproduced from the plates of H.O. 249. The tables were originally prepared at the U. S. Nautical Almanac Office of the U. S. Naval Observatory. The computations were performed on punched cards, and the printers' copy was prepared on a card-operated typewriter at the Naval Observatory. Photographic offset reproduction was used.

Volumes 2 and 3. These are permanent tables for the reduction of aeronautical observations of the Sun, Moon, and planets. The arguments are declination: 0°(1°)29°, and latitude: 0°(1°)89°. The third argument is local hour angle, at intervals of 1°(2° above latitude 70°). Volume 2 covers latitudes from 0° to 39°, volume 3 from 40° to 89°. Altitude and azimuth are given to the nearest minute of arc, and nearest degree, respectively. A column of differences of altitude is given for horizontal interpolation from one declination to the next higher one.

An interesting feature is the extension of the tabulated altitudes to negative values for the reduction of solar and lunar observations made at the high levels of flight of modern aircraft. A table of the Greenwich Hour Angle and Declination of the Sun for the years 1953–2000 is given at the back of each volume. Other tables, on the end papers, are similar to those of volume 1.

The method of preparation, using punched cards, was the same as for the first volume. Proofreading on all three volumes was facilitated by the comparison of cards punched from proof with those used in the preparation of printers' copy.

This helps to explain why the only errata listed are for figures not clearly printed, and which may be misread.

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45[Q].—SAMUEL HERRICK, "Tables for rocket and comet orbits," National Bureau of Standards AMS No. 20, U. S. Gov. Printing Office, Washington, 1953, xxiv + 100 p., 26.7 cm. Price $1.75.

In Dynamical Astronomy, and more recently in rocketry, the solution is frequently required of the equation, $E - e \sin E = M$ (or the corresponding equations for parabolic or hyperbolic motion). This is known as Kepler's Equation, and reputedly more than 300 papers have been written on this subject. Now comes another.

The justification of this one lies in the following considerations. When "e" is small, one encounters no difficulty in solving by successive approximations, using an ordinary table of sines. As "e" approaches unity, the number of decimals required near the front of the table becomes inordinately large in order to reach a comparable accuracy, since one essentially uses $(E - \sin E)$. The author has therefore chosen $(E - \sin E)$ as the argument of his tables, and the respondents are $(1 - \cos E)$, $\sin E$, and E (with corresponding functions for the other two types of motion). The difficulty with the decimal point cannot be overcome, but it is rendered less obtrusive by assigning "ranges" to the argument.

The Introduction is exceedingly lucid and detailed. The author emphasizes the relation of his concept to rectilinear motion, in contrast to the usual approach,

which is related to parabolic motion. Illustrations are given to match those of Bauschinger's classical astronomical treatise.

The tables are clearly printed by photo-offset from copy prepared on a punched-card controlled typewriter. Presumably there are no typographical errors, if this process is properly prosecuted. The actual calculations were carried out under the supervision of Dr. Gertrude Blanch, for the most part on an IBM 602 Multiplier. The interval of the argument varies throughout the table in such a fashion that interpolation is rendered uniform with first and modified second differences. Unfortunately, it was not considered feasible to print the one hundred entries of the Everett second difference coefficients on one page, or even on one opening.

The application to rockets, as the title imples, is not immediately apparent. The only times during which rockets are in rectilinear motion they are subjected to such strong forces of thrust or drag that the ordinary equations of two-body motion do not apply. The recently proposed artificial satellite will most certainly have an "e" less than 0.1, and even an intercontinental ballistic missile on any kind of a reasonable trajectory has a moderate eccentricity. If some future space cadet sets his rocket ship into rectilinear motion with respect to the Sun, he is surely doomed to perdition in a very real sense.

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46[S, T].—BEATRICE H. WORSLEY, "Solutions of a nonlinear differential equation arising in the theory of diffusion flames," *MTAC*, v. 9, 1955, p. 112–116.

The paper describes a numerical method of solving the equation

$$v'' = \frac{k}{x + x_1} v(v + x + x_1 - 1)$$

with the following two-point boundary conditions

$$v = 1 - x_1 \quad \text{at} \quad x = 0$$
$$v \to 0 \quad \text{as} \quad x \to \infty.$$

The computations were carried out with the aid of FERUT. Graphs and tables of $v(x)$ are available from the author for all the following cases:

$1 - x_1 = 0.01$ with $K = 0.001, 0.005, 0.01, 0.05, 0.1, 0.5, 1,$ and $5.$
$1 - x_1 = 0.2$ with $K = 0.001, 0.005, 0.01, 0.05, 0.1, 0.5, 1,$ and $5.$
$1 - x_1 = 0.4$ with $K = 0.005, 0.01, 0.05, 0.1, 0.5, 1, 5,$ and $10.$
$1 - x_1 = 0.6$ with $K = 0.01, 0.05, 0.1, 0.5, 1, 5,$ and $10.$
$1 - x_1 = 0.8$ with $K = 0.05, 0.1, 0.5, 1, 5, 10, 50,$ and $100.$
$1 - x_1 = 0.9$ with $K = 0.1, 0.5, 1, 5, 10, 50,$ and $100.$

A. H. T.

47[V].—STEFAN BERGMAN, "Tables for the determination of fundamental solutions of equations in the theory of compressible fluids," *MTAC*, v. 9, p. 8–14, 1955.

The tables referred to in the title are tables of the functions $F_n(\lambda)$, $T(\lambda)$ for complex values of the argument λ where

$$\lambda = \tfrac{1}{2} \log \left[\frac{1 - T}{1 + T} \left(\frac{1 + hT}{1 - hT} \right) \frac{1}{h} \right]$$

$$T = (1 - M^2)^{\frac{1}{2}}$$

$$h = \left(\frac{\gamma - 1}{\gamma + 1} \right)^{\frac{1}{2}}$$

$$F_n = (-1)^n \frac{(\gamma + 1)M^4}{64} \left[\frac{a_n M^4 + 4(3 - 2\gamma)M^2 - 16}{(1 - M^2)^{\frac{3}{2}}} \right], \quad n = 1, 2$$

$$a_1 = (3\gamma - 1), \quad a_2 = \gamma - 3;$$

γ is the ratio of specific heats presumably taken to satisfy $\gamma = 7/5$. The arguments and functional values are given to four decimal places.

A. H. T.

NOTES

Raymond Clare Archibald
1875–1955

Professor R. C. Archibald of Brown University, founder and editor of *Mathematical Tables and Other Aids to Computation*, died in Sackville, New Brunswick, July 26, 1955. He would have been 80 years of age on October 7.

R. C. A., as readers of *MTAC* knew him since 1943, had retired as editor in 1949. A portrait and an account of his life and his writings appeared at that time in *MTAC*, v. 4, 1950, p. 1–2. Since his retirement he had kept up an active interest in *MTAC* contributing articles and reviews of tables. His special interest in tables began with his studies of mathematics. In 1908 he commenced the collection of tables for the Brown University Library which was to become the most important collection of its kind. His connection with the National Research Council began in 1939 when he assumed the chairmanship of its *MTAC* committee. In a prophetic move, he saw the need of a periodical to further the exchange of information and ideas about tables and the art of computation. His high standards of scientific and bibliographic exactitude have been a challenge to the editors who have followed in his steps.

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Electronic Computer for the Indian Statistical Institute

The *Hindusthan Standard* on November 8, 1955 announced that the Indian Statistical Institute will receive an electronic computer from USSR. The Indian Statistical Institute is directed by Professor P. C. Mahalanobis.

Technical details of the computer were not given, but earlier reports by members of UNESCO committees indicated that USSR planned to deliver to India a copy of the machine built for the Academy of Sciences of the USSR. No report of this machine is known to exist in published scientific literature.

The USSR Academy of Sciences machine was at least partially described at the International Conference on Electronic Digital Computers and Information Processing held at Darmstadt, Germany, October 25–27, 1955 (announcement, *MTAC*, v. 9, 1955, p. 137). It has also been described in *Pravda*.

As it was described in Darmstadt and elsewhere, the machine seems to be a floating binary machine with electrostatic memory. The word length is 39 binary digits, it uses three address commands, it has approximately 2^{10} words of electrostatic storage, and a variety of other storage, some novel.

C. B. T.

Vychislitel'naîa Matematika i Vychislitel'naîa Tekhnika

We are glad to welcome a foreign competitor. The Institute for Exact Mechanics and Computational Technology of the Academy of Sciences of USSR began recently to issue (irregularly) a journal, *Vychislitel'naîa Matematika i Vychislitel'naîa Tekhnika*, that is, *Computational Mathematics and Computational Technology*. Apparently 5000 copies are printed. The first two numbers have reached this country. For the time being we propose to publish here a translation of the Table of Contents. In accordance with our policy, we do not intend to review the articles in numerical analysis but will review articles on tables and on other aids to computation, in particular on digital and analog computers. Reviews of the articles on numerical analysis are appearing in *Mathematical Reviews*.

The translation of the titles in Collection I has been made available by *Mathematical Reviews*; those of the titles in Collection II have been made by Ida Rhodes of National Bureau of Standards.

It is appropriate to call attention here to the useful sections on numerical and graphical methods and on calculating machines and instruments in the Russian reviewing journal, *Referativnye Zhurnal-Matematika*, which appeared first in October, 1953.

J. T.

Collection I, 1953

V. A. Ditkin i L. A. Lîusternik, Ob odnom prieme prakticheskogo garmonicheskogo analiza na sfere [On a method of practical harmonic analysis on the sphere]. 3

L. A. Lîusternik, Primenenie kubaturnykh formul k chislennomu resheniîu zadachi Koshi dlîa nekotorykh uravneniĭ matematicheskoĭ

fiziki [Application of cubature formulas to the numerical solution of Cauchy's problem for certain equations of mathematical physics]. 14

A. I. Ivanova, Nekotorye sluchai kubaturnoĭ formuly L. A. Lĭûsternika dlîâ pravil'nykh mnogougol'nikov [Certain cases of L. A. Lĭûsternik's cubature formula for regular polygons]. 27

A. A. Abramov, O vliĭânii oshibok okrugleniĭâ pri reshenii uravneniĭâ Laplasa [On the influence of round-off errors in the solution of LaPlace's equation]. 37

L. A. Lĭûsternik, O skhodimosti pri sluchaĭnykh nachal'nykh dannykh i nakoplenii oshibok iterafsionnogo profsessa resheniĭâ sistemy algebraicheskikh uravneniĭ [On the convergence and accumulation of errors of an iterative process of solution of a system of algebraic equations for random initial data]. 41

M. R. Shura-Bura, O reshenii konechno-raznostnogo uravneniĭâ, aproksimiruûshchego zadachu Dirikhle dlîâ uravneniĭâ Laplasa, na èlektricheskikh setkakh [On the solution by electric networks of a finite-difference equation approximating the Dirichlet problem for the Laplace equation]. 46

V. I. Shestakov, Modelirovanie operafsiĭ ischisleniĭâ predlozheniĭ posredstvom prosteĭshikh chetyrekhpolûsnykh skhem [Modelling the operations of the propositional calculus by means of the simplest four-pole networks]. 56

A. I. Vzorova, O reshenii sistemy lineĭnykh algebraicheskikh uravneniĭ sposobom IÛ. A. Shreĭdera [On the solution of a system of linear algebraic equations by the method of IÛ. A. Schreĭder]. 90–94

Collection II, 1955

G. S. Khovanskiĭ, Metodika postroeniĭâ nomogramm s orientirovannym transparantom [A method for the construction of nomograms with an oriented transparency]. 3

L. V. Bochek, O chislennom integrirovaniĭ uravnenii v kompleksnoĭ oblasti [On the numerical integration of equations in the complex domain]. 94

E. N. Dekanosidze, Nekotorye svoĭstva funkfsii Lommelîâ ot dvukh peremennykh [Certain properties of Lommel functions in two variables.] 97

L. N. Karmazina, Nekotorye svoĭstva korneĭ polinomov IÂkobi [Certain properties of the roots of Jacobi polynomials]. 108

L. N. Karmazina, Ob odnom sposobe vychisleniĭâ gipergeometricheskoĭ funkfsii [Concerning one method for the computation of the hypergeometric function]. 111

V. K. Saul'ev, K voprosu resheniĭâ zadachi o sobstvennykh znacheniĭâkh metodom konechnykh raznosteĭ [On the question of solving the eigenvalue problem by the method of finite differences]. 116

I. M. Stesin, Ofsenka tochnosti vychisleniĭâ sobstvennykh chisel apparatom nepreryvnykh drobeĭ [An estimate of the precision of the computation of eigenvalues by the means of continued fractions]. 145

M. A. Tomson, K voprosu ob issledovanii oblastei orbital'noĭ ustoĭchivosti [On the question of the investigation of regions of orbital stability]. 151

M. A. Tomson, Ob odnom sluchae issledovaniîa orbital'noĭ ustoĭchivosti reshenii sistemy differentsial'nykh uravneniĭ [On one case in the investigation of the orbital stability of solutions of systems of differential equations]. 209

L. I. Gutenmakher, G. K. Kuz'minok, L. S. Klabukova, Metod resheniîa lineĭnykh algebraicheskikh uravneniĭ na élektronno-lampovom integratore [A method of solving linear algebraic equations on a vacuum-tube integrator]. 230–246

CLASSIFICATION OF TABLES

A. Arithmetical Tables. Mathematical Constants

B. Powers

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